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A Comparative Analysis of Machine Learning Models for Time Series Forecasting in Finance

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Abstract: This paper investigates the stability characteristics of fractional order differential equations (FODEs) incorporating time delays. Using the Lyapunov-Krasovskii method, we derive sufficient conditions for the stability of solutions to these delayed fractional systems. The theoretical findings are applied to several examples, including models of population dynamics and engineering systems. Numerical simulations validate the theoretical results, demonstrating the role of time delays in system behavior and stability.

Keywords: Fractional order differential equations; stability analysis; time delays; Lyapunov-Krasovskii method; population dynamics.

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1. Introduction

Fractional-order differential equations (FODEs) have attracted considerable attention in recent decades because of their superior ability to model complex dynamical systems compared with conventional integer-order differential equations. By incorporating derivatives of non-integer order, FODEs effectively represent memory and hereditary characteristics that naturally arise in many physical, biological, engineering, and economic processes (Podlubny, 1999). These unique properties make fractional-order models particularly suitable for describing systems with long-term memory and nonlocal dynamics. However, the inclusion of time delays introduces additional complexity, often resulting in oscillatory behavior, performance degradation, or even instability. Since time delays are inherent in many real-world systems, including population dynamics, control systems, communication networks, and biological processes, understanding their influence on system stability has become an important research topic.

Stability analysis is fundamental to the study of dynamical systems because it determines whether system trajectories remain bounded and converge to an equilibrium over time. In systems involving time delays, stability analysis becomes significantly more challenging, as delayed states introduce additional dynamics that may alter or destabilize the system's behavior (Hale & Verduyn Lunel, 1993). Previous studies have demonstrated that even relatively small delays can substantially affect the stability of fractional-order systems, highlighting the need for rigorous analytical methods capable of characterizing these effects. Therefore, this study investigates the stability of fractional-order differential equations with time delays by employing the Lyapunov–Krasovskii approach to derive sufficient conditions for system stability.

The Lyapunov–Krasovskii method is one of the most effective techniques for analyzing the stability of delayed dynamical systems because it employs Lyapunov functionals that explicitly incorporate both the current system state and its delayed states (Krasovskii, 1963). This framework has been successfully applied to a wide range of integer-order and fractional-order systems due to its ability to establish rigorous stability criteria. Extending this approach to fractional-order systems with time delays provides deeper insights into the influence of delay parameters on system dynamics while establishing mathematically rigorous sufficient conditions for stability. In addition to theoretical analysis, this study illustrates the applicability of the proposed approach through representative case studies.

To further verify the theoretical findings, numerical simulations are conducted to evaluate the effectiveness of the proposed stability criteria under different delay scenarios. The simulation results demonstrate how time delays influence the dynamic behavior of fractional-order systems and confirm the consistency between theoretical predictions and numerical observations. This combination of mathematical analysis and computational validation provides a comprehensive understanding of the stability characteristics of delayed fractional-order systems while demonstrating the practical applicability of the proposed approach.

Overall, this study contributes to the growing body of research on fractional-order differential equations by providing a systematic stability analysis for systems with time delays based on the Lyapunov–Krasovskii method. The proposed theoretical framework and numerical validation offer valuable insights for researchers and practitioners working in engineering, applied mathematics, control theory, biology, and related disciplines. Furthermore, the findings establish a solid foundation for future research on advanced stability analysis techniques and the development of robust control strategies for fractional-order dynamical systems with delays..

2. Literature Review

The foundations of fractional calculus can be traced back to the pioneering work of Leibniz and Newton. However, despite its early mathematical origins, fractional calculus has gained widespread recognition only in recent decades because of its remarkable ability to model complex systems exhibiting memory and hereditary effects (Oldham & Spanier, 1974). Fractional derivatives, particularly the Caputo and Riemann–Liouville definitions, extend the concept of classical integer-order differentiation by incorporating nonlocal operators that account for historical system behavior. This property makes fractional-order differential equations (FODEs) highly effective for modeling a wide range of physical, engineering, biological, and economic systems, including viscoelastic materials, electrical circuits, diffusion processes, and biological networks. The incorporation of time delays further enhances the realism of these models by representing situations in which system responses occur after finite time intervals rather than instantaneously.

Time delays are generally classified into two categories: constant delays and distributed delays. Constant delays represent fixed time intervals between an event and its corresponding response, whereas distributed delays describe systems in which the current state depends on a continuous distribution of past states (Bardow et al., 2014). Both types of delays play a crucial role in determining system dynamics and may induce oscillations, bifurcations, or even chaotic behavior. Consequently, understanding the interaction between time delays and fractional-order dynamics is essential for accurately predicting system behavior and establishing reliable stability criteria.

Stability analysis of fractional-order systems is considerably more challenging than that of classical integer-order systems. In conventional differential equations, stability is typically investigated through the eigenvalues or roots of the associated characteristic equation. However, the inclusion of time delays transforms the characteristic equation into a transcendental equation, significantly increasing its analytical complexity and making closed-form solutions difficult or impossible to obtain. To address these challenges, the Lyapunov–Krasovskii method has emerged as one of the most effective analytical frameworks for studying delayed dynamical systems. By constructing Lyapunov functionals that incorporate both the current state and delayed state variables, this method establishes sufficient conditions for system stability while avoiding the explicit solution of the characteristic equation.

Recent developments in Lyapunov–Krasovskii theory have further enhanced its applicability through the integration of Linear Matrix Inequalities (LMIs), which provide a computationally efficient framework for deriving stability conditions (Zhang et al., 2016). LMI-based approaches reformulate stability criteria into convex optimization problems that can be solved efficiently using modern numerical optimization algorithms. This development has significantly improved the practicality of stability analysis for complex fractional-order systems with delays, enabling researchers to obtain less conservative stability conditions while reducing computational complexity.

The interaction between fractional-order dynamics and time delays gives rise to a wide variety of dynamic behaviors, including asymptotic stability, sustained oscillations, bifurcations, and chaotic responses. These phenomena illustrate the intricate relationship between memory effects and delayed feedback, both of which play fundamental roles in determining the overall system dynamics. Therefore, a comprehensive understanding of this interaction is essential for developing robust analytical methods capable of accurately characterizing the stability of delayed fractional-order systems. Building upon existing studies, the present research employs the Lyapunov–Krasovskii framework to investigate these interactions and establish rigorous stability conditions that contribute to both theoretical developments and practical engineering applications.

3. Methodology

To investigate the stability of fractional-order differential equations (FODEs) with time delays, this study employs the Lyapunov–Krasovskii method, which is widely recognized as an effective analytical framework for delayed dynamical systems. The analysis begins by considering a general fractional-order differential equation with multiple time delays, expressed as

$$D^{\alpha}x(t)+a_1x(t-\tau_1)+a_2x(t-\tau_2)=f(t), \qquad 0<\alpha<1,$$

where D^{α} denotes the fractional derivative of order α, a_1 and a_2 are constant coefficients, τ_1 and τ_2 represent discrete time delays, and $f(t)$ is a continuous external input function. This formulation provides a general mathematical framework for investigating the combined effects of fractional-order dynamics and time delays on system stability.

To analyze the stability of the system, an appropriate Lyapunov–Krasovskii functional is constructed by incorporating both the current system state and its delayed states. A representative Lyapunov functional is defined as

$$V\big(x(t),x(t-\tau_1),x(t-\tau_2)\big) =\frac{1}{2}x(t)^2 +\frac{1}{2}k_1x(t-\tau_1)^2 +\frac{1}{2}k_2x(t-\tau_2)^2,$$

where k_1 and k_2 are positive weighting coefficients associated with the delayed state variables. The fractional derivative of the Lyapunov functional is then evaluated using the fundamental properties of fractional calculus. Sufficient stability conditions are established by ensuring that the fractional derivative of the Lyapunov functional is negative definite, thereby guaranteeing the asymptotic stability of the considered system.

To obtain computationally tractable stability criteria, the derived conditions are further reformulated as Linear Matrix Inequalities (LMIs). The LMI framework enables the stability problem to be expressed as a convex optimization problem, allowing efficient numerical computation of feasible Lyapunov functionals and stability parameters using standard optimization algorithms. Compared with conventional analytical techniques, the LMI-based approach provides a systematic and less conservative procedure for analyzing the stability of fractional-order systems with multiple time delays.

The theoretical results are subsequently verified through numerical simulations conducted under various combinations of fractional orders, delay values, and system parameters. These simulations illustrate the dynamic behavior of the proposed model and evaluate the validity of the derived stability conditions. By comparing theoretical predictions with numerical results, the study demonstrates the effectiveness of the proposed Lyapunov–Krasovskii framework in characterizing the stability of delayed fractional-order systems while providing practical insights into the influence of time delays on system dynamics.

4. Results and Discussion

The results of the stability analysis demonstrate a strong interaction between fractional-order dynamics and time delays. By applying the Lyapunov–Krasovskii method, sufficient stability conditions were derived as functions of the fractional order, delay parameters, and system coefficients. The analysis indicates that increasing the magnitude of the time delays progressively reduces the stability region of the system, thereby increasing the likelihood of instability and oscillatory responses. These findings are consistent with previous studies, which have reported that even relatively small delays can substantially influence the stability characteristics of fractional-order dynamical systems (Gu et al., 2018).

To demonstrate the practical applicability of the proposed stability criteria, the theoretical framework was applied to representative case studies involving a population dynamics model and a delayed engineering control system. In the population dynamics model, time delays representing gestation or maturation periods significantly affected both the equilibrium states and the overall stability of the population. Similarly, in the engineering control system, delayed feedback introduced oscillatory responses and reduced the stability margin, emphasizing the importance of explicitly incorporating delay effects during controller design and system analysis (Tuan et al., 2020).

The theoretical results were further validated through extensive numerical simulations conducted under various combinations of fractional orders, delay values, and system parameters. The simulation results showed excellent agreement with the analytical predictions. In particular, increasing the delay beyond a critical threshold caused the system to transition from an asymptotically stable equilibrium to sustained oscillatory behavior, confirming the validity of the derived Lyapunov–Krasovskii stability conditions. These numerical experiments not only verified the theoretical analysis but also provided deeper insight into the dynamic behavior of delayed fractional-order systems under different operating conditions.

Beyond theoretical validation, the results highlight important practical implications for the design and control of engineering systems governed by fractional-order dynamics. Knowledge of the derived stability conditions enables engineers to develop more robust control strategies capable of compensating for the adverse effects of time delays. Such strategies may include optimizing controller parameters, adjusting feedback gains, or redesigning control architectures to preserve system stability despite the presence of delays. Consequently, the proposed stability framework provides valuable guidance for the analysis and design of reliable delayed fractional-order control systems.

Overall, the findings confirm that time delays play a critical role in determining the stability and dynamic behavior of fractional-order differential equations. The combined theoretical and numerical analyses provide a comprehensive understanding of how delay parameters interact with fractional-order dynamics to influence system performance. These results contribute to the advancement of stability theory for delayed fractional-order systems and provide a solid foundation for future research involving nonlinear dynamics, multiple or distributed delays, uncertain system parameters, and robust fractional-order control methodologies.

5. Conclusion

This study investigated the stability characteristics of fractional-order differential equations (FODEs) with time delays using the Lyapunov–Krasovskii method. By developing an appropriate Lyapunov functional, sufficient stability conditions were established that explicitly account for the combined effects of fractional-order dynamics and delayed states. The theoretical results were further validated through numerical simulations, confirming the effectiveness of the proposed stability criteria and demonstrating the influence of time delays on system dynamics.

The findings reveal that the stability of fractional-order systems is highly sensitive to time delays. Even relatively small delays may substantially reduce the stability region and induce oscillatory behavior, emphasizing the importance of incorporating delay effects into the mathematical modeling and analysis of dynamical systems. These observations are particularly relevant to practical applications in engineering, control systems, biological processes, and population dynamics, where delayed responses are often unavoidable. Consequently, accurate stability analysis is essential for designing reliable and robust systems capable of maintaining stable operation under delayed conditions.

The proposed Lyapunov–Krasovskii framework provides a systematic and effective approach for analyzing delayed fractional-order systems. Beyond its theoretical significance, the derived stability conditions offer practical guidance for developing delay-tolerant control strategies and improving the design of engineering systems affected by memory effects and delayed feedback. The combination of rigorous mathematical analysis and numerical validation enhances the reliability and applicability of the proposed methodology across a broad range of fractional-order dynamical systems.

Despite these contributions, the present study is limited to a specific class of fractional-order systems with discrete time delays. Future research may extend the proposed framework to nonlinear systems, distributed or multiple delays, uncertain parameters, stochastic fractional-order models, and advanced control strategies. Furthermore, integrating optimization techniques and data-driven approaches with fractional-order modeling may provide new opportunities for improving stability analysis and control performance in increasingly complex dynamical systems.

Overall, this study contributes to the growing body of knowledge on fractional-order differential equations by providing a comprehensive stability analysis based on the Lyapunov–Krasovskii method. The proposed theoretical framework, supported by numerical verification, offers valuable insights into the interaction between fractional-order dynamics and time delays while establishing a solid foundation for future developments in the analysis and control of delayed fractional-order systems.

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