International Journal of Applied Mathematics and Computing
Vol. 2, No.1 January 2025
e-ISSN: 3047-146X; Page 01-14
DOI ; https://doi.org/10.62951/ijamc.v2i1.33
Available online at: https://international.arimsi.or.id/index.php/IJAMC

On Fuzzy $\hat{\alpha}g$-Topological R-Module Spaces

Basim Mohammed Melgat
AL-Qasim Green University College of Education Dep. of Mathematic, Babylon, Iraq
Email ; dr.basimmohammed@wrec.uoqasim.edu.iq

Abstract: *The aim of this research is to give a new definition and study a new notion of fuzzy α̂g-topological R-module spaces using fuzzy α̂g-open sets. The concept of fuzzy α̂g-topological R-modules is introduced and study them to provide a mathematical basis. Traditional cautiousness for F bisected by α̂g-tautologies reigns furtive-space yield we assay different fuzzy α̂g-separation axioms such as* $\hat{\alpha}g - T_{F_i}$*; where* $i = \{0,1,2,3\}$ *in the context of these spaces. Also, we study fuzzy* $\hat{\alpha}g$*-separation axiom* $\hat{\alpha}g - T_{F_i}$*,* $i = 0,1,2,3$ *in fuzzy* $\hat{\alpha}g$ *- Topological. R-Module space. The results indicate that throughout the domain of a fuzzy topological R-module space* (W, τ_{F_W}) *the following fuzzy separation axioms* $\hat{\alpha}g - T_{F_0}$, $\hat{\alpha}g - T_{F_1}$, $\hat{\alpha}g - T_{F_2}$, *and* $\hat{\alpha}g - T_{F_3}$ *are valid . Also, we also show that possessing fuzzy α̂g-open neighborhoods has further consequences to the structure and continuity of fuzzy topological spaces. The main conclusion of our research is that the mysterious α̂g topological R-unit spaces are a new and interesting area for further study. The terms of their operations in addition to providing important information about the interaction between fuzziness and topological properties that may be useful in many mathematical and applied branches.*

Keyword: *Fuzzy* $\hat{\alpha}g$ *- Topological. R-Module Space, Fuzzy* $\hat{\alpha}g$*-* T_{F_i} *Spaces,* $i = 0,1,2,3$

1. INTRODUCTION AND PRELIMINARIES:

The purpose of the study is to define the notation of fuzzy $\hat{\alpha}g$ - Topological. R-Module (fz. $\hat{\alpha}g$ - Topo. R-Mod.). in the term of fz set. The fz concept has used in many branches of mathematics since 1965 The fuzziness given by (Zadeh, 1965). the fuzziness of Topo. space studied by (Chang, 1968 and Lowen 1976). The theories of fz Topo. mod. and fz Topo. sub mod. given by (Al-Shamiri, 2020) . fuzziness of bi- Topo. R-mod. given by (Melgat, 2023) and (Melgat, 2024). We consider it in terms of fuzzy $\hat{\alpha}g$ -open set (fz $\hat{\alpha}g$ -open) set and. fz separation axiom $\hat{\alpha}g - T_{F_i}$, $i = 0,1,2,3$ by (Imran, 2015). A subset N of a Topo. space W is called fz $\hat{\alpha}g$ -open if $(cl(N^o))^o \leq M$ whenever $N \leq M$ and M is fz open in W, while N^c is a fz $\hat{\alpha}g$ -closed in W in fz $\hat{\alpha}g$-Topo. R-mod

Defi. 1.1.1 [10]

A map $\varphi: W \to J$ ($J = \{0 \leq w \leq 1\}$) is a fz. subset M of W, that is an element of J^W,we referred by $M(w)$ or w_α to the belonging of w in M .

This definition offers a mathematical way to define fuzzy sets mathematically in a space by assigning a value between 0 and 1 to each component of the (W) set , showing the exact membership level of the element to the fuzzy subset.

Defi. 1.1.2[1]

Received: September 15, 2024 Revised: September 28, 2024 Accepted: Oktober 31, 2024 Online Available: November 02, 2024

A class $\tau_F \in J^W$ of fz set is a fz Topo. for W if the next statements are hold:

1) $1_\emptyset, 1_W \in \tau_{F_W}$
2) $\forall M, N \in \tau_{F_W} \rightarrow M \wedge N \in \tau_{F_W}$
3) $\forall (M_j)_{j \in J} \in \tau_{F_W} \rightarrow \vee_{j \in J} M_j \in \tau_{F_W}$

(W, τ_{F_W}) is called fz. Topo. space. M is fz open if $M \in \tau_{F_W}$. and M^c is a fz. closed set.

This definition describes the shape of a fuzzy topology and shows how fuzzy sets can be taken and mixed through intersection and union and retain their status as belonging to the fuzzy topology.

Defi. 1.1.3 [7]

let W be a left R-mod. A fz set M in W is called a fz. left R-mod. if for each $w, v \in W$ and $h \in R$:

(1) $M(w + v) \geq min\{M(w), M(v)\}$.
(2) $M(w) = M(w^{-1})$.
(3) $M(h.w) \geq M(w)$.
(4) $M(0) = 1$.

This definition sets the stage to discuss the Fuzzy left R-modules and underlines the properties which form the relationship between algebraic structures and fuzzy set theory. It finds a non-decreasing membership function for addition, symmetry of inverse element and for scalar multiplication one must be more appropriate for modeling the real-life uncertainty. On this way, by defining $(M(0) = 1)$ it gives the minimum potential of certainty and provides clear references for the decisions with the use of fuzzy sets and fuzzy arithmetic, as well as a clear foundation for the fuzzy optimization and systems analysis. This integration with traditional algebra sustains theoretical work and real-life applications in environments where uncertainty is present.

Defi 1.1.4 [7]

let M be a left R-mod., the set W is said to be left fz Topo R-mod. if:

(1) W left fz R-mod.
(2) W is a fz Topo group on W and satisfies the following axiom:

The mapping $\varphi: R \times W \longrightarrow W$ Defined by $\varphi(r, w) = r.w, (r \in R \text{ and } w \in W)$ is a fz cont.

This definition therefore combines the algebraic structure of fuzzy left (R) −modules together with the topological structure needed for the fuzzy analysis.

Defi. 1.1.5 [9]

A fz. Topo. space $(\mathrm{W},\tau_{F_W})$is said to be fz $\hat{\alpha}g\ - T_{F_0}$-Topo. space iff $\forall w, v \in W, w \neq v, \exists\ \hat{\alpha}g\ - M \in \tau_{F_W}$ s.t either $\hat{\alpha}g\ - M(w) =\ 1$ and $\hat{\alpha}g\ - M(v) = 0$ or $\hat{\alpha}g\ - M(w) =\ 0$ and $\hat{\alpha}g\ - M(v) = 1$.

This definition introduces a certain kind of separation property for the fuzzy points in the space showing that how the different points could be recognized by means of fuzzy membership functions.

Defi. 1.1.6 [9]

We said to be fz $\hat{\alpha}g\ - T_{F_1}$-Topo. space of (W,τ_{F_W}) iff $\forall w, v \in W, w \neq v, \exists\ \hat{\alpha}g\ - M \in \tau_{F_W}$ s.t $\hat{\alpha}g\ - M(w) =\ 1$ and $\hat{\alpha}g\ - M(v) = 0$ and $\hat{\alpha}g\ - M(w) =\ 0$ and $\hat{\alpha}g\ - M(v) = 1$.

This definition demonstrates that for each two different elements of the set (W) there always exist fuzzy sets that can separate one element from the other, which proves the drudges of the fuzzy topology. It virtually means that the structure enables the depiction of local differences of points in the space through fuzzy sets.

Defi. 1.1.7 [9]

A fz Topo. space (W,τ_{F_W}) is called a fz $\hat{\alpha}g\ - T_{F_2}$-Topo. space iff for any two fz. points $w \neq v \in W$, there is $\hat{\alpha}g\ - M, \hat{\alpha}g\ - N \in \tau_{F_W}$with $\hat{\alpha}g\ - M(w) = \hat{\alpha}g\ - N(v) = 1$ and $\hat{\alpha}g\ - M \wedge \hat{\alpha}g\ - N = 1_{\emptyset}$

This has been expressed in the following definition: It is possible to select two fuzzy sets for a fuzzy $\hat{\alpha}g$ -Topological space such that each point of the fuzzy set is represented exclusively by its own fuzzy set and has no overlap with the other. This enhances expressiveness of the fuzzy topology where the focus is given to the sort generalizations of the finest distinctions of points in a space.

Defi. 1.1.9 [9]

(W,τ_{F_W}) called fz $\hat{\alpha}g$ -regular if $\forall w \in W$ and each fz closed set $\hat{\alpha}g\ - H$ such that $\hat{\alpha}g - H(w) = 0$ there are fz $\hat{\alpha}g$ -open sets $\hat{\alpha}g\ - M$ and $\hat{\alpha}g\ - N$ s.t $\hat{\alpha}g\ - M(w) > 0$, $\hat{\alpha}g\ - H \leq \hat{\alpha}g\ - N$and $\hat{\alpha}g$ -$M \wedge \hat{\alpha}g\ - N = 1_{\emptyset}$.

This definition gives an understanding, in fuzzy $\hat{\alpha}g$-regular space, of any clutter point that is not part of the fuzzy closed set, one can separate it from the set using fuzzy open sets that hold the degree of membership at that point for the fuzzy set and which are disjoint from the closed fuzzy set. This property is crucial for setting certain desirable attributes of the topology, as one may understand the similar term 'regulation' in traditional topological spaces.

Defi 1.1.10 [9]

A fz Topo. space (W, τ_{F_W})is said to be fz $\hat{\alpha}g - T_{F_3}$-Topo. space iff it is fz. $\hat{\alpha}g - T_{F_1}$-Topo. space and fz $\hat{\alpha}g$ − regular.

From this definition it can be seen that a fuzzy $\hat{\alpha}g - T_{F_3}$-topological space is obtained by adding the separtion axioms of a fuzzy $\hat{\alpha}g - T_{F_1}$-space with the regularity condition .Thus, it is a higher order of fuzzy topological space in which R involves the fuzzy open set which can separate two arbitrary point while also making sure that there exist a fuzzy closed set to separate a given point from any other. These together strengthen the framework of the topological space and complement the application of the concept in the fuzzy topology.

Thm. 1.1.12 [3]

For any fz Topo. R-mod. space(M, τ_{FM}), the next conds. are true

1) (M, τ_{FM}) is fz T_{F_2} Topo. R-mod. space
2) $\{0_\alpha\}$ is fz closed subset in M.
3) If $\{U_0\}$ is a basis of nbhd of 0_α, then $\bigcap_{V\in U_0} V = \{0_\alpha\}$
4) (M, τ_{FM}) is fz T_{F_0}
5) (M, τ_{FM}) is fz T_{F_1}
6) (M, τ_{FM}) is fz T_{F_3}

This theorem forms the basis of analyzing the propositions related to the properties of fuzzy topological R-modules. The rationale is that we have been able to reduce the problem to the assertion that such spaces have to be separated a priori by virtue of certain properties. The membership functions are: T_{F_0}, T_{F_1},T_{F_2}, and T_{F_3}; and the operations include the fuzzy closed set at the zero element of the set space. Taken together, each of these conditions serves to enhance the investigation of the structure and dynamics of such fuzzy realms.

2. Fz $\hat{\alpha}g$. Topo. R-mod. Spaces

Defi. 2.2.1

A fz. set $M \subset W$ is called fz $\hat{\alpha}g$ -open nbhd for $w \in W$ if there is a fz $\hat{\alpha}g$ -open set N with $w \in N \leq M$.

This definition establishes the basic structure of fuzzy neighborhoods, enabling us to ensure the existence of an open fuzzy set containing the sample point (w) and thus enhancing the local topological structure of the fuzzy space.

Defi. 2.2.2

A function $\hat{\alpha}g - \varphi: (W, \tau_{F_W}) \to (V, \tau_{F_V})$ is called fz $\hat{\alpha}g$-conts. if $\forall \hat{\alpha}g - M \leq V \Longrightarrow \hat{\alpha}g - \varphi^{-1}(M) \leq W$.

This condition is essential for continuity concerns in the fuzzy environment, especially to ensure that initial images are mapped to fuzzy sets within the original space. It complements traditional continuity while at the same time responding to the innate lack of clarity associated with the structure.

Defi. 2.2.3

A function $\hat{\alpha}g - \varphi: (W, \tau_{F_W}) \to (V, \tau_{F_V})$ is fz $\hat{\alpha}g$ -open function if $\forall \hat{\alpha}g - M \leq W \Longrightarrow \hat{\alpha}g - \varphi(M) \leq V$. Moreover, the notion of $\hat{\alpha}g - \varphi$ it's bijective, both $\hat{\alpha}g - \varphi$ and $\hat{\alpha}g - \varphi^{-1}$are fz $\hat{\alpha}g - cont.$

This definition shows a higher degree of structure for functions between fuzzy spaces so that maps preserve the open fuzzy structure. The emergence of thematic requirements amplifies the utility of function in presenting it as a symmetrical mapping between ambiguous topologies that would numb the audience to potential interactions between these spaces.

Defi. 2.2.4

A pair (W, τ_{F_W}) , where W is R- mod. and τ_{F_W} a Topo. on W , is called a $\hat{\alpha}g$. Topo. R-mod. if (W, τ_R) $fz.topo.ring$ and the following maps is fz. $\hat{\alpha}g$-cont.

$\hat{\alpha}g - \varphi: R \times W \to W, \hat{\alpha}g - \varphi(k, w) \to k.w \ , w \in W \ and \ k \in R$

This construct combines algebraic and topological aspects: we shall use it to illustrate how modules can be fuzzified. Also, the continuity of scalar multiplication in this sense suggests that the new fuzzy topological space gets the functional nature expected of R-modules alongside with fuzzy notions.

Thm. 2.2.5

Let (W, τ_{F_W}) is a fz $\hat{\alpha}g$ -Topo. R-mod., $w \in W,\ r \in R$ s.t $w_\alpha \in \{v: W(v) = \max\{W(h)\}, \forall h \in W\}$, then

$a)\ \hat{\alpha}g - \varphi_k: (W, \tau_{F_W}) \to (W, \tau_{F_W}), \hat{\alpha}g - \varphi_{r_\beta}(w) = k.w,\ k \in W$ is a fz $\hat{\alpha}g$ -cont. mapping

$b)\ \hat{\alpha}g - \varphi_w: (R, \tau_{F_R}) \to (W, \tau_{F_W}),\ \ \hat{\alpha}g - \varphi_{w_\beta}(w) = k.w,\ k \in R$ is fz $\hat{\alpha}g$-cont. mapping

This theorem reassures the stability of the fuzzy α̂g-topological R-modules that scalar multiplications are smoothly included in the fuzzy structure. This opens the door for large applications and especially in the analysis and functional equations within fuzzy environments.

Proof

Result from Defi. of fz Topo-R-mod. 2.2.4

Corollary 2.2.6

Let (R, τ_R) is a fz $\hat{\alpha}g$-Topo. abelian ring with a identity and (W, τ_{F_W}) is fz $\hat{\alpha}g$-Topo. R-mod. Let $w \in W$ has an inverse element, $w_\alpha \in \{v: W(v) = \max\{W(h)\}, \forall h \in W\}$ and $r \in R$, so

$a)\ \hat{\alpha}g - \varphi_k: (W, \tau_{F_W}) \to (W, \tau_{F_W}), \hat{\alpha}g - \varphi_k(w) = k.w,\ k \in W$ is a fz $\hat{\alpha}g$-homeo.

$b)\ \hat{\alpha}g - \varphi_w: (R, \tau_{F_R}) \to (W, \tau_{F_W}),\ \ \hat{\alpha}g - \varphi_{w_\beta}(w) = k.w, k \in R$ is a fz $\hat{\alpha}g$-homeo.

This result gives the relationship between algebra and topology in fuzzy situation thus supports the fact that ever element of inverse works predictability in structure of fuzzy R-modules.

Thm. 2.2.7

Suppose (W, τ_W) a fz $\hat{\alpha}g$-Topo-R-mod. and let $\{\hat{\alpha}g - M_0\}$ be a fund. Sys. of fz $\hat{\alpha}g$-nbhds of 0 such that $M(0) = \max\{W(v)\},\ \forall v \in W\}$, if $\hat{\alpha}g - M \in \hat{\alpha}g - M_0\ and\ \hat{\alpha}g - N \in \hat{\alpha}g - M_0$ then the next statements are true

1) $\hat{\alpha}g - M \wedge \hat{\alpha}g - N \in \{\hat{\alpha}g - M_0\}$

2) $-(\hat{\alpha}g - M) \in \{\hat{\alpha}g - M_0\}$.

3) $\hat{\alpha}g - M$ is symmetric

4) $\hat{\alpha}g - M + \hat{\alpha}g\text{-}N \in \{\hat{\alpha}g - M_0\}$

5) $(\hat{\alpha}g - M).(\hat{\alpha}g\text{-}N) \in \{\hat{\alpha}g - M_0\}$

6) for a fz $\hat{\alpha}g$ −nbhd $\hat{\alpha}g - H$ of 0 in R then $\hat{\alpha}g - H.\hat{\alpha}g - N \leq \hat{\alpha}g - M$.

7) for any $\hat{\alpha}g - M \in \{\hat{\alpha}g - M_0\}$and $r \in R$ there exist $\hat{\alpha}g - N \in \{\hat{\alpha}g - M_0\}$ s.t $r.\hat{\alpha}g - N \leq \hat{\alpha}g - M$.

8) for any $\hat{\alpha}g - M \in \hat{\alpha}g - M_0$ and $w \in W$ there exist a fz nbhd $\hat{\alpha}g - N$ of 0 in R s.t $\hat{\alpha}g - N.m \leq \hat{\alpha}g - M$.

Stating the following facts as the theorem about fuzzy neighborhoods, they proclaim that fuzzy neighborhoods follow a predetermined pattern similar to the usual topological neighborhoods and strengthen the foundation of fuzzy arithmetic to guarantee the stability of calculations. These results serve for controlling the fuzzy topological structures in the applications and subsequent investigations.

Proof

(1)

Let $\hat{\alpha}g - M \in \{\hat{\alpha}g - M_0\}$ and $N \in \{\hat{\alpha}g - M_0\}$, then there exists $\hat{\alpha}g - \rho_1, \hat{\alpha}g - \rho_2 \in \tau_{F_R}$ s.t $\hat{\alpha}g - \rho_1(0) = \hat{\alpha}g - M(0) > 0$, $\hat{\alpha}g - \rho_1 \leq \hat{\alpha}g - M$ and $\hat{\alpha}g - \rho_2(0) = \hat{\alpha}g - N(0) > 0$, $\hat{\alpha}g - \rho_2 \leq \hat{\alpha}g - N$. We have $\hat{\alpha}g - \rho_1 \wedge \hat{\alpha}g - \rho_2 \in \tau_{F_R}$ and $\hat{\alpha}g - \rho_1 \wedge \hat{\alpha}g - \rho_2 \leq \hat{\alpha}g - M_1 \wedge \hat{\alpha}g - N$

Also $(\hat{\alpha}g - \rho_1 \wedge \hat{\alpha}g - \rho_2)(0) = \min\{\hat{\alpha}g - \rho_1(0), \hat{\alpha}g - \rho_2(0)\} = \min\{\hat{\alpha}g - M(0), \hat{\alpha}g - N(0)\}$

Thus $\hat{\alpha}g - M \wedge \hat{\alpha}g - N \in \{U_0\}$

The properties that emerge from the proof (1) reveal a well-defined fuzzy neighborhood in the fuzzy α-g topological R-module(W, τ_W). More specifically, they emphasize the local character of the sum and product, the corresponding homologies in the algebraic structures under addition and multiplication, and the zone continuity under scalar multiplication and local tuning. This foundational framework enables the analyses of continuation and joining in this fuzzy topological environment.

(2)

Let $\hat{\alpha}g - M \in \{\hat{\alpha}g - M_0\}$, then there exists $\hat{\alpha}g - \rho \in \mu$ s.t $\hat{\alpha}g - \rho(0) = \hat{\alpha}g - M(0) > 0$,$\hat{\alpha}g - \rho \leq \hat{\alpha}g - M$. Let $\hat{\alpha}g - \varphi: R \to R$, $\hat{\alpha}g - \varphi(r_\alpha) = -r_\alpha$, then $\hat{\alpha}g - \varphi$ is a fz. $\hat{\alpha}g$ −homeo. (Corollary 2.2.6) implies $\hat{\alpha}g - g^{-1}(\hat{\alpha}g - \rho) \in \tau_{F_R}$. Also

$\hat{\alpha}g - g^{-1}(\hat{\alpha}g - \rho) = \hat{\alpha}g - \rho(\hat{\alpha}g - g(r)) \leq \hat{\alpha}g - M(\hat{\alpha}g - g(r)) = \hat{\alpha}g - M(-r) = -(\hat{\alpha}g - M)(r)$ and $-(\hat{\alpha}g - M)(0) = \hat{\alpha}g - M(-0) = \hat{\alpha}g - M(0) > 0$ Thus

$-(\hat{\alpha}g - M) \in \{\hat{\alpha}g - M_0\}$

The second proof states that in the context of fuzzy topological R-Mod., the fuzzy neighborhood holds the closure and symmetry in the different operations to add a powerful framework for studying the algebraic as well as the topological properties of those fuzzy neighborhoods. Especially, it illustrates how structure enables mathematical computation while keeping properties of neighborhood intact to provide better understanding of the implied topology of the space.

(3)

Let $\hat{\alpha}g - M \in \{\hat{\alpha}g - M_0\}$, then by condition (2) we have $-(\hat{\alpha}g - M) \in \{\hat{\alpha}g - M_0\}$, then there exists $\hat{\alpha}g - \rho \in \tau_{F_R}$ s.t $\hat{\alpha}g - \rho_1(0) = \hat{\alpha}g - M(0) > 0$,$\hat{\alpha}g - \rho_1 \leq \hat{\alpha}g - M$ and $-(\hat{\alpha}g - \rho_1) \leq -(\hat{\alpha}g - M)$. let $\hat{\alpha}g - \rho = \hat{\alpha}g - \rho_1 \wedge -(\hat{\alpha}g - \rho_1)$, then $\hat{\alpha}g - \rho = -(\hat{\alpha}g - \rho)$ is symmetric. Also $-(\hat{\alpha}g - \rho)(r) = \min\{\hat{\alpha}g - \rho_1(r), -(\hat{\alpha}g - \rho_1)(r)\} \leq \hat{\alpha}g - M(r)$ And $-(\hat{\alpha}g - \rho)(0) = \min\{\hat{\alpha}g - \rho(0), -(\hat{\alpha}g - \rho)(0)\} = \min\{\hat{\alpha}g - M(0), -(\hat{\alpha}g - M)(0)\} =$ $\min\{\hat{\alpha}g - M(0), \hat{\alpha}g - M(-0)\} = \min\{\hat{\alpha}g - M(0), \hat{\alpha}g - M(0)\} = \hat{\alpha}g - M(0)$Thus $\hat{\alpha}g - M$ is symmetric

By showing that for any fuzzy neighborhood$(\hat{\alpha}g - M)$ in the fundamental system $\{\hat{\alpha}g - M_0\}$ of fuzzy neighborhoods of zero the set$\{\hat{\alpha}g - M_0\}$ belongs to the same system, it proves the symmetry of fuzzy neighborhoods.This symmetry is useful to ensure that for every fuzzy neighborhood of zero, it has the corresponding negative neighborhood, hence making an enhancement to the structures of the fuzzy topological R-modules and operations on them.

(4)

Let $\hat{\alpha}g - M$ be an fz. $\hat{\alpha}g$ −nbhd of $0, \hat{\alpha}g - M(0) > 0$, then there exists $\hat{\alpha}g - \mathrm{N} \in \hat{\alpha}g - \tau_{F_F}$, $\hat{\alpha}g - \mathrm{N}(0) > 0$ s.t $\hat{\alpha}g - \mathrm{N}(0) \leq \hat{\alpha}g - \mathrm{M}(0)$. let $\hat{\alpha}g - \varphi: R \times R \to R$; $\hat{\alpha}g - \varphi(r_\alpha, k_\alpha) = r_\alpha + k_\alpha$, Since $(R, \hat{\alpha}g - \tau_{F_R})$ is a fz $\hat{\alpha}g -$ Topo. ring, then the map $\hat{\alpha}g - \varphi$ is fz. $\hat{\alpha}g$ −cont. and therefore, $\hat{\alpha}g - \varphi^{-1}(\hat{\alpha}g - N)$ is fz. $\hat{\alpha}g$ −open in $(R \times R, \hat{\alpha}g - \tau_{F_R} \times \hat{\alpha}g - \tau_{F_R})$. But $\hat{\alpha}g - \varphi^{-1}(\hat{\alpha}g - N)(0,0) = \hat{\alpha}g - N(\hat{\alpha}g - g(0,0)) = \hat{\alpha}g - \mathrm{N}(0.0) = \hat{\alpha}g - \mathrm{N}(0) > 0$. Hence, there exists fz. $\hat{\alpha}g$ −open set $\hat{\alpha}g - N_1$ with $\hat{\alpha}g - N_1 \times \hat{\alpha}g - N_1 \subset g^{-1}(\hat{\alpha}g - N)$ s.t $(\hat{\alpha}g - N_1 \times \hat{\alpha}g - N_1)(0,0) > 0$. Thus $\hat{\alpha}g - \varphi(\hat{\alpha}g - N_1 \times \hat{\alpha}g - N_1) = \hat{\alpha}g - N_1 + \hat{\alpha}g - N_1 \leq (\hat{\alpha}g - N) + (\hat{\alpha}g - N) \leq \hat{\alpha}g - M + \hat{\alpha}g - M$ implies $(\hat{\alpha}g - M) + (\hat{\alpha}g - M) \in \{\hat{\alpha}g - M_0\}$

The proof (4) proves that the range αg-topology in the R-module $\hat{\alpha}g - M$ is fuzzy and has closure characteristics with addition and multiplication, the symmetry and supports all operation in the scalar multiplication. These properties altogether confirm that the construction of fuzzy αg-neighborhoods makes sense in the sense that the structure of a fuzzy topological R-module is preserved for the operations. These results offer an important background for proceeding and extending research into fuzzy topology and its applications to other mathematical structures.

(5) By the same way of (4)

(6)

Let $\hat{\alpha}g - M$ be an fz. $\hat{\alpha}g$ −nbhd of 0 and $\hat{\alpha}g - M(0) > 0$, then $\exists \hat{\alpha}g - \mathrm{N} \in \tau_{F_W}$, $\hat{\alpha}g - \mathrm{N}(0) > 0$, $\hat{\alpha}g - \mathrm{N} \leq \hat{\alpha}g - M$ and $\hat{\alpha}g - N(0) = \hat{\alpha}g - M(0) = \max\{W(h)\}, \forall h \in W\}$. Since (W, τ_{F_W}) is a fz. $\hat{\alpha}g$ −Topo-R-mod. space, there is a fz. $\hat{\alpha}g - \mathrm{H}$ of 0 in R and fz. $\hat{\alpha}g$ −open set $\hat{\alpha}g - N_1 \in \tau_{F_W}$, (by Corollary 2.2.6) $\hat{\alpha}g - \varphi(\hat{\alpha}g - H \times \hat{\alpha}g - N) \leq \hat{\alpha}g - \mathrm{H}.\hat{\alpha}g - N \leq \hat{\alpha}g - H.\hat{\alpha}g - M$

Proof (6) proved that operations like intersection, addition scalar multiplication, and forming complements of fuzzy $\hat{\alpha}g -$ topological R-module space(W, τ_{F_W}) with a fundamental system of fuzzy $\hat{\alpha}g$ −neighborhoods $\hat{\alpha}g - M(0)$ around zero are closed. This illustrates a rigid and robust classification model of fuzzy neighborhoods that can consistently regain their properties even when undergoing operations and can locally fluctuate around points in the space.

(7)

Let $\hat{\alpha}g - M$ be an fz. $\hat{\alpha}g$ −nbhd of 0 and $\hat{\alpha}g - M(0) > 0$, then $\exists \hat{\alpha}g - N \in \tau_{F_W}$, $\hat{\alpha}g - N(0) > 0$ s.t $\hat{\alpha}g - N \leq \hat{\alpha}g - M$ and $\hat{\alpha}g - N(0) = \hat{\alpha}g - M(0) = \max\{W(h)\}$, $\forall h \in W\}$. let $\hat{\alpha}g - \varphi_k : R \rightarrow M$; $\hat{\alpha}g - \varphi_k(w) = k.w$, Since (W, τ_{F_W}) is a fz. $\hat{\alpha}g -$ Topo-R-mod. space, then the mapping$v$ is fz. $\hat{\alpha}g -$ homeo. [corollary 2.2.6] and therefore $r.\hat{\alpha}g - N$ is fz. $\hat{\alpha}g -$ open set of 0 [corollary 2.2.6]. Hence, there exists fz. $\hat{\alpha}g -$ open set $\hat{\alpha}g - N_1$ with $\hat{\alpha}g - N_1 \leq \hat{\alpha}g - {\varphi_k}^{-1}(r.\hat{\alpha}g - N_1)$. Thus $k.\hat{\alpha}g - N_1 \leq k.\hat{\alpha}g - N \leq k.\hat{\alpha}g - M$

According to proof (7), in a fuzzy (\hat{g})-topological R-module (W, τ_{F_W}) with a fundament system of fuzzy $\hat{\alpha}g$ −neighborhoods $\hat{\alpha}g - M(0)$ around (0), it can be observed that falling with closure of intersection and addition, symmetry, and compatible with scalar multiplication .Furthermore, it guarantees the existence of fuzzy neighborhoods similar to arbitrary fuzzy neighborhoods centered at 0 and provides a dense structure that supports both algebraic computations and topological separations within the module, which in turn forms the foundation of the theoretical study of fuzzy topological spaces.

(8)

Let $\hat{\alpha}g - M$ be an fz. $\hat{\alpha}g$ −nbhd of 0 and $\hat{\alpha}g - M(0) > 0$, then $\exists \hat{\alpha}g - N \in \tau_M$, $\hat{\alpha}g - N(0) > 0$ s.t $\hat{\alpha}g - N \leq \hat{\alpha}g - M$ and $\hat{\alpha}g - N(0) = \hat{\alpha}g - M(0) = \max\{W(h)\}$, $\forall h \in W\}$. let $\hat{\alpha}g - \varphi_w : R \to W$; $\hat{\alpha}g - \varphi_w(k) = k.w$, Since $(W, \tau_{F_W})$is a fz. $\hat{\alpha}g$ −Topo-R-mod. space, then the mapping $\hat{\alpha}g - \varphi_w$ is fz $\hat{\alpha}g$ −homeo. [corollary 2.2.6] and therefore $\hat{\alpha}g - \mathrm{N}.w$ is fz. $\hat{\alpha}g$ −open set of 0[corollary 2.2.6]. Hence, there exists fz. $\hat{\alpha}g$ −open set $\hat{\alpha}g - N_1$ with $\hat{\alpha}g - N_1 \leq \hat{\alpha}g - \varphi_w{}^{-1}(r.\hat{\alpha}g - N)$. Thus $\hat{\alpha}g - N_1.w \leq \hat{\alpha}g - N.w \leq \hat{\alpha}g - M.w$

According to proof (8), in a fuzzy $\hat{\alpha}g$ -topological R-module, the family of fuzzy (\alpha)-neighborhoods of 0 that is $\hat{\alpha}g - M(0)$ is closed under finite intersection, addition and scalar multiplications. Each fuzzy neighborhood is symmetric, and all the fuzzy products of neighborhoods must bound while providing robust local topological structure to the module that enables the continuity of mappings.

3. Fz $\hat{\alpha}g$ -T_i-Spaces, $i = 0, 1, 2, 3$

Defi. 3.3.1

A fz $\hat{\alpha}g$-Topo. R-mod. space $(W, \hat{\alpha}g - \tau_{F_W})$is said to be fz $\hat{\alpha}g - T_{F_0}$ $\hat{\alpha}g$ Topo. R-mod. space iff $\forall w, v \in \hat{\alpha}g - W, w \neq v, \exists$ $\hat{\alpha}g - M \in \hat{\alpha}g - \tau_{F_W}$s.t either $\hat{\alpha}g - M(w) = 1$ and $\hat{\alpha}g - M(v) = 0$ or $\hat{\alpha}g - M(v) = 1$and $\hat{\alpha}g - M(w) = 0$.

This formulation defines the Fuzzy Separation Property and we are able to separate two different points using a fuzzy set.

Example 3.3.2

Let $\hat{\alpha}g - \varphi: \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}_2$ by $\hat{\alpha}g - \varphi(\mathrm{n}, \mathrm{w}) = \mathrm{nw}$ for all $n \in \mathbb{Z}$ and $w \in \mathbb{Z}_2$ (integers modulo 2),i.e $\varphi(\mathrm{n}, \mathrm{w}) = \mathrm{w} + \mathrm{w} + \cdots + \mathrm{w}$ (n-times). Let a fz set $\hat{\alpha}g - M_1, \hat{\alpha}g - M_2$ on $\mathbb{Z}_2$ as

$\hat{\alpha}g - M_1([0]) = 1, \hat{\alpha}g - M_1([1]) = 0$,

$\hat{\alpha}g - M_2([0]) = 0 , \hat{\alpha}g - M_2([1]) = 1$

for all $w \in \mathbb{Z}_2$. Let $\hat{\alpha}g - \tau_{F_W} = \{\emptyset, \mathbb{Z}_2 \hat{\alpha}g - M_1, \hat{\alpha}g - M_2\}$ is a fz $\hat{\alpha}g$ −Topo. R-mod. on $\mathbb{Z}_2$, then $(\mathbb{Z}_2, \hat{\alpha}g - \tau_{F_W})$ is a fz $\hat{\alpha}g - \mathrm{T}_{F_0}$ Topo. R-mod. space

Using these sets, the set $\hat{\alpha}g - \varphi(\mathrm{n}, \mathrm{w}) = \mathrm{nw}$ forms a fuzzy R-topological α̂g-module on$\mathbb{Z}_2$. Hence, $(\mathbb{Z}_2, \hat{\alpha}g - \tau_{F_W})$ corresponds to a fuzzy module space α̂g-T_{F_0} topo. R, which shows that the points can be clearly identified.

Defi. 3.3.3

A fz $\hat{\alpha}g$-Topo. R-mod. space $(W, \hat{\alpha}g - \tau_{F_W})$is said to be fz $\hat{\alpha}g - T_{F_1}$ $\hat{\alpha}g$ Topo. R-mod. space iff $\forall w, v \in \hat{\alpha}g - W, w \neq v, \exists$ $\hat{\alpha}g - M, \hat{\alpha}g - N \in \hat{\alpha}g - \tau_{F_W}$s.t $\hat{\alpha}g - M(w) = 1$ and $\hat{\alpha}g - M(v) = 0$ or $\hat{\alpha}g - N(v) = 1$and $\hat{\alpha}g - M(w) = 0$.

Fuzzy $\hat{\alpha}g - T_{F_1}$ spaces further develop the fuzzy $\hat{\alpha}g - \mathrm{T}_{F_0}$ spaces where one can associate two different fuzzy sets with the same point characterizing it differently. This established a richer structure, while still classifying points in a fuzzy manner, thus enabling better feature-wise robustness.

Example 3.3.4

Let $\hat{\alpha}g - \varphi: \mathbb{Z}_2 \times \mathbb{Z} \rightarrow \mathbb{Z}_2$ by $\hat{\alpha}g - \varphi(\mathrm{n}, \mathrm{w}) = \mathrm{nw}$ for all $n \in \mathbb{Z}$ and $w \in \mathbb{Z}_2$ (integers modulo 2),i.e $\varphi(\mathrm{n}, \mathrm{w}) = \mathrm{w} + \mathrm{w} + \cdots + \mathrm{w}$ (n-times). Let a fz set $\hat{\alpha}g - M_1, \hat{\alpha}g - M_2$ on $\mathbb{Z}_2$ as

$\hat{\alpha}g - M_1([0]) = 0.5, \hat{\alpha}g - M_1([1]) = 0$,

$\hat{\alpha}g - M_2([0]) = 0$, $\hat{\alpha}g - M_2([1]) = 0.5$

$\hat{\alpha}g - M_3([0]) = 0.5$, $\hat{\alpha}g - M_3([1]) = 0.5$

Let $\hat{\alpha}g - \tau_{F_W} = \{\emptyset, \mathbb{Z}_2, \hat{\alpha}g - M_1, \hat{\alpha}g - M_2, \hat{\alpha}g - M_3\}$ is a fz $\hat{\alpha}g$ –Topo. R-mod. on $\mathbb{Z}_2$, then $(\mathbb{Z}_2, \hat{\alpha}g - \tau_{F_W})$ is a fz $\hat{\alpha}g - \mathrm{T}_{F_1}$ $\hat{\alpha}g$ –Topo. R-mod. space

An example with $\mathbb{Z}_2$ has demonstrated that the space exists in this structure of fuzzy $\hat{\alpha}g - T_{F_1}$ by utilizing mixed values of fuzzy sets $\hat{\alpha}g - M_1, \hat{\alpha}g - M_2, \hat{\alpha}g - M_3$. This case confirms and illustrates how fuzzy spaces $\hat{\alpha}g - T_{F_1}$ operate with fuzzy properties that allow fine-grained membership gradations rather than binary classifications, which are essential features in many applications that are considered fuzzy.

Defi. 3.3.5

A fz $\hat{\alpha}g$ –Topo. R-mod. space $(W, \hat{\alpha}g - \tau_{F_W})$ is said to be fz $\hat{\alpha}g - T_{F_2}$ Topo. R-mod. space iff for $w \neq v \in W$, there exists $\hat{\alpha}g - M, \hat{\alpha}g - N \in \hat{\alpha}g - \tau_{F_W}$ with $\hat{\alpha}g - M(w) = \hat{\alpha}g - N(v) = 1and\hat{\alpha}g - M \wedge \hat{\alpha}g - N = \emptyset$

Fuzzy $\hat{\alpha}g - T_{F_2}$ adds more meaning to the degree of separation by making arrangements of guaranteeing that the differentiating set between (w) and (v) is disjoint. This aspect enhances the topological structure by ensuring that some feature points are relatively isolated, suitable for applications where partitioning is important.

Example 3.3.6

Let $\mathbb{Z}_4$ (integers modulo 4) be $\mathbb{Z}$ −mod. Define $\hat{\alpha}g-\varphi:\mathbb{Z}_2\times\mathbb{Z}\to\mathbb{Z}_2$ by $\hat{\alpha}g-\varphi(\mathrm{n},\mathrm{w})=\mathrm{nw}$ for all $n\in\mathbb{Z}$ and $w\in\mathbb{Z}_2$,i.e $\hat{\alpha}g-\varphi(\mathrm{n},\mathrm{w})=\mathrm{w}+\mathrm{w}+\cdots+\mathrm{w}$ (n-times) with fz discrete Topo. on it,

then $(\mathbb{Z}_4,\tau_{FD})$ is a fz $\hat{\alpha}g-\mathrm{T}_{F_2}$ $\hat{\alpha}g$ −Topo. R-mod. space

If you look at the structure of $\mathbb{Z}_4$, you can easily distinguish how all the fuzzy $\hat{\alpha}g-\mathrm{T}_{F_2}$ properties are fulfilled through mapping and the fuzzy separation. This example can serve as a concrete validation of the theoretical construct of fuzzy $\hat{\alpha}g-\mathrm{T}_{F_2}$ space and identify proper uses of fuzzy definitions in problem solving which improve the capability of the space to separate points.

Defi. 3.3.7

A fz $\hat{\alpha}g$ −Topo. R-mod. space $(W,\hat{\alpha}g-\tau_{F_W})$ will be called fz $\hat{\alpha}g$ −regular $\hat{\alpha}g$ −Topo. R-mod. space if for each fz point $w\in W$ and each fz $\hat{\alpha}g$ −closed set H such that $H(w)=0$ there are fz $\hat{\alpha}g$ −open sets M and N such that $M(w)>0$, $H\leq N$ and $M\wedge N=\emptyset$

The new idea of fuzzy regularity strengthens the structure by introducing constraints on the connections between points and closed sets. It guarantees the existence of a neighborhood of such a point such that the habitat of the fuzzy topology always remains outside a closed set and hence we have a well-framed structure of the fuzzy topological structure.

Defi. 3.3.8

A fz $\hat{\alpha}g$ −Topo R-mod. $(W,\hat{\alpha}g-\tau_{F_W})$ is said to be fz $\hat{\alpha}g-T_{F_3}$ $\hat{\alpha}g$ −Topo. R-mod. space if $(W,\hat{\alpha}g-\tau_{F_W})$ is fz $\hat{\alpha}g-T_1$ $\hat{\alpha}g$ −Topo R-mod. space and fz $\hat{\alpha}g$ −regular $\hat{\alpha}g$ −Topo. R-mod. space.

The extension of the $\hat{\alpha}g-T_1$ property in conjunction with $\hat{\alpha}g$ −regularity delivers a sound background to the fuzzy topological spaces and more greatly promising them in satisfying the level of separation and structure in analysis and in applications.

Example 3.3.9

Let $\mathbb{R}$ be (real space) be $\mathbb{Z}$ −mod. define $\hat{\alpha}g-\varphi:\mathbb{R}\times\mathbb{Z}\to\mathbb{R}$ by $\hat{\alpha}g-\varphi(\mathrm{n},\mathrm{r})=\mathrm{nr}$ for all $n\in\mathbb{Z}$ and $r\in\mathbb{R}$,i.e $\hat{\alpha}g-\varphi(\mathrm{n},\mathrm{r})=\mathrm{r}+\mathrm{r}+\cdots+\mathrm{r}$ (n-times) with fz usual $\hat{\alpha}g$ −Topo. $\hat{\alpha}g-\tau_{F_U}$ on it. Then $(\mathbb{R},\hat{\alpha}g-\tau_{FU})$ is fz $\hat{\alpha}g-T_{F_3}$ $\hat{\alpha}g$ −Topo. R-mod. space.

In the case of the given example using the real space (R), specification of the fuzzy$\hat{\alpha}g - T_{F_3}$ structure is shown with the help of mapping and fuzzy topology as specified above. It gives the understanding of effective application of fuzzy logic in traditional topological frameworks inherent in many widely used frameworks By means of exemplification of the fuzzy $\hat{\alpha}g - T_{F_3}$as in a familiar mathematical space.

Theorem 3.3.10

Every fz $\hat{\alpha}g - T_{F_i}$ is fz $\hat{\alpha}g - T_{F_{i-1}}$, $i = 0{,}1.2{,}3$

This theorem postulates that: every ‘fuzzy structure ‘contains the properties of the ‘fuzzy $\hat{\alpha}g - T_i$ for (i = 0, 1, 2, 3), This creates a layering hierarchy among the T_i spaces and boosts the stability of the ‘fuzzy septem separation axioms .’This theory is fundamental, and refers to the fact that stronger separating properties extend from weaker properties thus developing related systems to explain ambiguous topological actions. In other words, this result confirms the setting of the above definitions in a clear and orderly manner, which enables a systematic analysis of the properties of fuzzy α̂g-topological spaces. This gradation suggests that as we go higher in the classes, the rich topological features are helpful in creating an understanding of fuzziness within topology.

Proof: Clearly.

4. CONCLUSIONS:

1. The newly defined fuzzy $\hat{\alpha}g$ −topological spaces have different properties from the usual topological fuzzy spaces and traditional R −modules. This is illustrated by one of the most promising and recent trends, which is dealing with classical topological concepts in ambiguous environments.
2. Promising results have been obtained regarding the axioms of fuzzy $\hat{\alpha}g - T_{F_W}$. These results not only provide a further description of fuzzy topological spaces$\hat{\alpha}g - T_{F_i}$ but also provide an exploration of further separability properties in the fuzzy environment.
3. The findings of this research can be considered as generalizations of existing theories in the field of fuzzy topology. It suggests that many classical results can be reconceptualized in a fuzzy context without compromising the basic principles of topological spaces.

4. The implications of this research are manifold, expanding potential applications into areas such as fuzzy logic, decision making, and mathematical modeling, where inherent uncertainty and indeterminacy prevail. Future research may explore more complex interactions within various structures of fuzzy$\hat{\alpha}g$ −topological spaces, as well as deeper applications in real-world scenarios.

5. REFERENCES

Al-Shamiri, M. M. A. (2020). Some results on fuzzy topological modules and fuzzy topological submodules. *Asian Journal of Fuzzy and Applied Mathematics*, 8(1).

Chang, C. L., & Fz Topo. (1968). Topological spaces. *Journal of Mathematical Analysis and Applications*, 24, 182–190.

Imran, Q. H., & Abdulkadhim, M. M. (2015). Fuzzy separation axioms via fuzzy α̂g-kernel sets in fuzzy topological spaces. *European Journal of Scientific Research*, 128(3), 223–232.

Melgat, B. M., & Al-Khafaji, M. (2019). Fuzzy neighborhood systems in fuzzy topological rings. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, 11(4), 87–93.

Melgat, B. M., & Al-Khafaji, M. (2019). Fuzzy topological rings induced by fuzzy pseudo normed rings. *Journal of Advanced Research in Dynamical and Control Systems*, 11(7), Special Issue.

Melgat, B. M., & Al-Khafaji, M. (2019). Some results of fuzzy separation axioms in fuzzy topological ring space. *MJPS*, 6(2).

Palaniappan, N. (2002). *Fuzzy topology*. Narosa Publications.

Ray, A. D. (2011). On (left) fuzzy topological modules. *International Mathematics*, 6(25–28), 1303–1312.

Ray, A. D., & Chettri, P. (2009). On fuzzy topological modulated valued fuzzy continuous functions. *Applied Mathematical Sciences*, 3(24), 1177–1188.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.