International Journal of Applied Mathematics and Computing
E-ISSN: 3047-146X
P-ISSN: 3048-1988

Using the Aquila Optimizer to Estimate the Two Parameters of the Fréchet Distribution

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Abstract: The Fréchet distribution is one of the commonly used Extreme Value Distributions (EVDs) in statistical modeling and heavy-tailed data analysis, where it plays an important role in describing product lifetimes as well as climatic and financial phenomena. The estimation of its two parameters, namely the shape parameter and the scale parameter, is traditionally based on the Maximum Likelihood Estimation (MLE) method. However, maximizing the likelihood function for this distribution involves numerical difficulties, which necessitates the use of numerical optimization methods. In this study, we propose the use of the Aquila Optimizer (AO), a recent metaheuristic algorithm inspired by the hunting behavior of eagles, as an efficient numerical tool for maximizing the likelihood function of the Fréchet distribution. The objective function was formulated as the negative log-likelihood function (-LogL), and the Aquila Optimizer was employed to obtain the optimal estimates of the distribution parameters. Several simulation experiments with different sample sizes were conducted to compare the performance of the proposed method with a conventional approach represented by the Nelder–Mead method, using the Mean Squared Error (MSE) criterion. The simulation results demonstrated that the Aquila Optimizer outperformed the Nelder–Mead algorithm in many cases, although the superiority was slight. The results also showed that both algorithms were consistent, as their MSE values decreased with increasing sample size. In addition, a practical application was carried out using real data, and the results of the survival function estimation indicated a good fit.

Keywords: Aquila Optimizer; Fréchet Distribution; Maximum Likelihood Estimationm; Metaheuristic Optimization; Simulation Study.

Received: October 18, 2025;
Revised: December 20, 2025;
Accepted: February 27, 2026;
Publish: April 11, 2026;
Curr. Ver.: April 11, 2026.

1. Introduction

Extreme Value Analysis is considered one of the fundamental topics in applied statistics because it plays a crucial role in describing rare phenomena or extreme events such as natural disasters, floods, mechanical failures, and financial crises. Among the probability distributions that have attracted considerable attention in this field is the Fréchet distribution, which represents one of the three models of extreme value distributions according to the Fisher–Tippett–Gnedenko theorem. This distribution is characterized by its ability to model heavy-tailed data, making it particularly suitable for the analysis of failure and reliability data, as well as environmental and economic data (Kotz & Nadarajah, 2000).

The estimation of the two parameters of the Fréchet distribution, namely the shape parameter and the scale parameter, is an essential step toward obtaining an accurate statistical description. Traditional studies in this area have relied primarily on the. Maximum Likelihood Estimation (MLE) method because of its desirable properties in terms of efficiency and consistency. However, the application of this method faces practical challenges when maximizing the likelihood function, especially with small or complex datasets, where issues may arise due to the absence of closed-form solutions and the difficulty of reaching optimal values using conventional numerical optimization algorithms (Alotaibi et al., 2024). To address these challenges, recent studies have increasingly turned to integrating metaheuristic algorithms as alternative or supportive tools in optimization processes. These algorithms, inspired by nature and mathematical principles, have demonstrated their effectiveness in overcoming traditional limitations and providing stable and flexible solutions in complex search spaces (Al-Sinjary, 2025). Among these algorithms, the Aquila Optimizer (AO) has recently emerged as a promising method inspired by the hunting behavior of eagles. AO is distinguished by its ability to balance exploration and exploitation within the search space, thereby enhancing the likelihood of reaching global optimal solutions rather than becoming trapped in local optima (Abualigah et al., 2021).

In this research, we aim to employ the Aquila Optimizer as a numerical tool for maximizing the likelihood function of the Fréchet distribution and, consequently, obtaining accurate estimates of its parameters. This will be achieved by formulating the objective function based on the negative log-likelihood function (-LogL) and then applying the AO algorithm to extract the optimal values. The study will also be supported by simulation experiments using samples of different sizes, in addition to a practical application to real data, with the results of AO being compared with those of traditional methods in terms of accuracy, stability, and convergence speed.

2. Fréchet Distribution

The Fréchet distribution (Fr) was first introduced by the French mathematician Maurice Fréchet. It is a continuous probability distribution with a heavy tail and is also considered a special case of the Generalized Extreme Value (GEV) distribution. This distribution is widely used in failure analysis, climatic data analysis (such as maximum wind speed and extreme floods), and economic data involving extreme values. If the random variable follows a Fréchet distribution, then its probability density function is given as follows (Musa & Elnoor; Araheemah & Al-Sarraf):

$$f(x;\alpha,\beta)=\begin{cases}\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}} & x\,,\alpha\,,\beta>0\\ 0 & o.w\end{cases} \quad \ldots(1)$$

Where α represents the shape parameter and β represents the scale parameter.

The Cumulative Distribution Function (CDF), Survival Function, and Hazard Function of the Fréchet distribution can also be defined as follows:

$$F(x;\alpha,\beta)=e^{-(\frac{\beta}{x})^{\alpha}} \qquad x>0 \qquad ;\ \alpha,\beta>0 \qquad \ldots(2)$$

$$S(x)=1-F(x)=1-e^{-\left(\frac{\beta}{x}\right)^{\alpha}} \qquad \ldots(3)$$

$$h(x)=\frac{f(x)}{s(x)}=\frac{\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}}{1-e^{-(\frac{\beta}{x})^{\alpha}}} \qquad \ldots(4)$$

The following figures illustrate the graphs of these four functions for different parameter val ues (Al-Salmany, 2021).

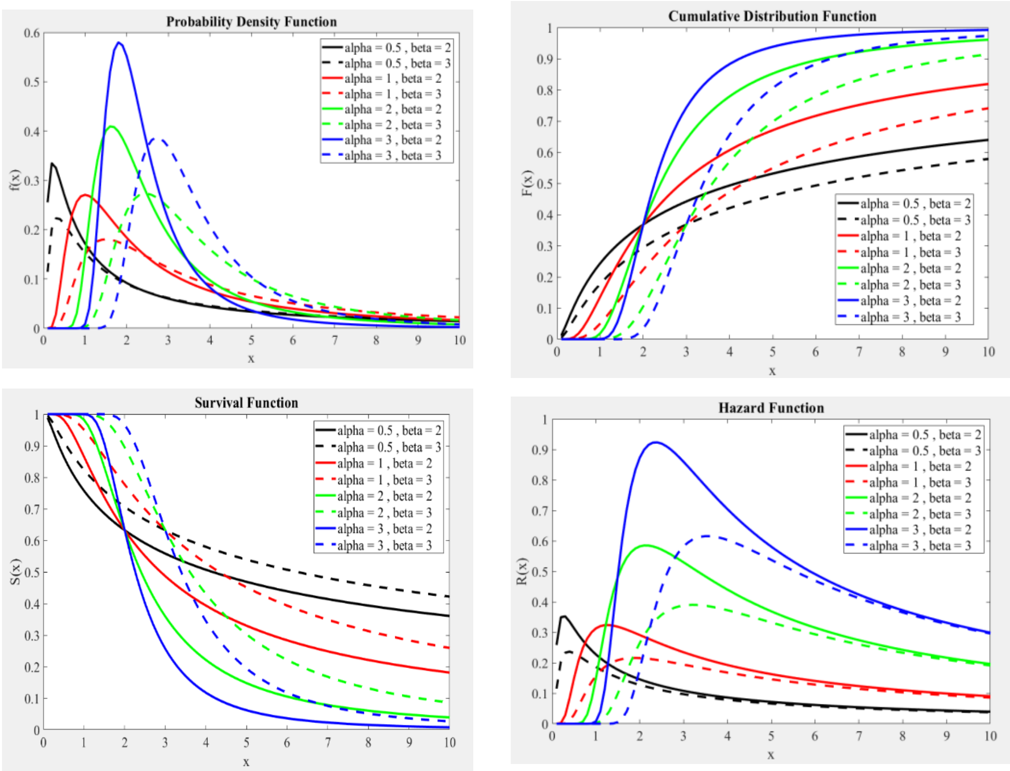

Figure 1. Probability Density Function, Cumulative Distribution Function, Survival Function, and Hazard Function of the Fr distribution.

The quantile function of this distribution can also be defined as follows:

$$Q(u) = F^{-1}(u)$$

$$Q(u) = \frac{\beta}{(-\ln u)^{\frac{1}{\alpha}}} \qquad \dots(5)$$

Given n observations from a random sample drawn from the Fr distribution, the likelihood function is given by:

$$L(\alpha,\beta) = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} {x_i}^{-(\alpha+1)} \right) \; e^{-\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}} \qquad \dots(6)$$

By taking the natural logarithm of both sides of Equation (6) above, we obtain:

$$l(\alpha,\beta) = n\,ln\,\alpha + n\alpha\,ln\,\beta - (\alpha+1)\sum_{i=1}^{n} ln\,x_i - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha} \qquad \dots(7)$$

By maximizing the log-likelihood function defined in Equation (7), the maximum likelihood estimates are obtained according to the optimization algorithm employed.

3. Nelder–Mead Algorithm

This algorithm was introduced by John Nelder and Roger Mead in their well-known paper, A Simplex Algorithm for Function Minimization (Nelder & Mead, 1965). The Nelder–Mead (NM) algorithm is considered one of the most widely used derivative-free optimization algorithms in statistical applications because of its ability to handle nonlinear functions without requiring the calculation of partial derivatives.

The algorithm is based on the concept of a simplex, which is a set consisting of (p + 1) points in a p-dimensional space. It operates by updating this simplex through four main operations: reflection, expansion, contraction, and shrinkage. These operations aim to achieve a balance between exploration and exploitation within the search space, allowing the algorithm to converge toward the optimal solution without being directly trapped in suboptimal local points. A summary of this method is presented below (Takenaga, 2023; Galántai, 2024).

Let $f(\psi)$be the function to be minimized, where $\psi$is the parameter vector containing p parameters; that is:

$$\psi = [\psi_1, \psi_2, \dots, \psi_p]$$

An initial simplex consisting of (p + 1) points is then constructed; that is:

$$\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(p)}, \psi^{(p+1)}$$

After evaluating the objective function (the function to be minimized) at each point, the points are ordered such that:

$$f(\psi^{(1)}) \le f(\psi^{(2)}) \le \cdots \le f(\psi^{(p)}) \le f(\psi^{(p+1)})$$

where:

$\psi^{(1)}$: the best point.

$\psi^{(p+1)}$: the worst point.

After that, the following steps are carried out:

Compute the centroid

$$\overline{\psi} = \frac{1}{p}\sum_{i=1}^{p} \psi^{(i)} \qquad \dots (8)$$

Reflection

$$\psi_r = \overline{\psi} + r(\overline{\psi} - \psi^{(p+1)}) \qquad \dots (9)$$

$$f_r = f(\psi_r) \qquad \dots (10)$$

where r denotes the reflection coefficient.

Expansion

If $f(\psi_r) < f(\psi^{(1)})$

$$\psi_e = \overline{\psi} + e(\psi_r - \overline{\psi}) \qquad \dots (11)$$

$$f_e = f(\psi_e) \qquad \dots (12)$$

where e denotes the expansion coefficient.

The better point is then selected from between ψ_r and ψ_e, according to which yields the smaller value of the objective function.

Contraction

If $f(\psi_r) \ge f(\psi^{(p)})$

We compute both:

Outside contraction: if $f(\psi^{(p)}) \le f(\psi_r) < f(\psi^{(p+1)})$

$$\psi_c = \overline{\psi} + c(\psi_r - \overline{\psi}) \qquad \dots (13)$$

Inside contraction: if $f(\psi_r) \ge f(\psi^{(p+1)})$

$$\psi_c = \overline{\psi} - c(\psi_r - \overline{\psi}) \qquad \dots (14)$$

where c denotes the contraction coefficient.

Shrink

If the contraction step fails:

$\psi^{(i)} = \psi^{(1)} + s\left(\psi^{(i)} - \psi^{(1)}\right), for\ i = 2,3, \dots, p+1$ ... (15)

where s denotes the shrinkage coefficient.

The stopping criterion is either reaching the maximum number of iterations or satisfying the following condition:

$max_i \left\| f\left(\psi^{(i)}\right) - f\left(\psi^{(1)}\right) \right\| < \varepsilon$

In the R statistical environment, this algorithm is implemented through the optim function using the "Nelder-Mead" option (R Core Team, 2025). It should be noted that this function uses the following parameters by default:

$r = 1, e = 2, c = 0.5, s = 0.5$

4. Aquila Optimizer Algorithm

The Aquila Optimizer (AO) is considered one of the modern metaheuristic algorithms. It was developed in 2021 by Abualigah et al. (Abualigah et al., 2021). This algorithm is based on simulating the hunting behavior of the eagle, including searching for prey, monitoring it, swooping down on it, and finally capturing and controlling it. The algorithm is distinguished by its ability to achieve an effective balance between exploration and exploitation, which makes it suitable for solving complex nonlinear optimization problems, as is the case in maximizing likelihood functions in statistical models.

The algorithm updates the positions of candidate solutions within the search space through four main strategies: two for exploration and two for exploitation. The transition among these strategies depends on the progress of time (iterations), thereby ensuring a gradual shift from broad search to precise search around the optimal solution. The mathematical steps of the algorithm are presented as follows:

Let $X(t)$denote the position of the eagle at iteration t, and let $X_{\text{best}}(t)$denote the best solution found so far. The updating process is then carried out according to the following cases:

High soaring and circling

$X(t+1) = X_{best}(t)\left(1 - \frac{t}{T}\right) + (U - L)R_1$ ... (16)

Attack from high position

$X(t+1) = X_{best}(t) + F\left(R_2 X_{best}(t) - X(t)\right)$ ... (17)

Stooping attack

$X(t+1) = X_{best}(t) + S\ sin(R_3)|R_4 X_{best}(t) - X(t)|$ ... (18)

Grabbing prey

$X(t+1) = X_{best}(t) + S\ cos(R_3)|R_4 X_{best}(t) - X(t)|$ ... (19)

where:

t: the current iteration number.

L, U: the lower and upper bounds of the search space.

T: the total number of iterations.

R_1, R_2, R_3, R_4: random variables following a uniform distribution.

F, S : dynamic control parameters that facilitate the transition from exploration to exploitation; usually, this transition is governed by the following function:

$$S = 2\left(1 - \frac{t}{T}\right)$$

This function gradually reduces exploration and increases exploitation as the iterations progress.

It is worth noting that, in each iteration, one of the four equations is used according to the value of $t/T$and certain random variables. Moreover, like the Nelder–Mead algorithm, the Aquila Optimizer does not require the computation of derivatives of the objective function, which makes it suitable for likelihood maximization problems that may be non-differentiable. Since no ready-made function for the Aquila Optimizer is available in the R language, the researcher implemented the algorithm manually.

5. Simulation

In this section, a comparison will be made between the Nelder–Mead (NM) algorithm and the Aquila Optimizer (AO) algorithm for maximizing the likelihood function of the Fréchet distribution defined in Equation (6). The data will be generated from the Fréchet distribution based on the inverse function defined in Equation (5).The data generation process will be carried out for eight different cases, as presented in Table (1), and for each case, different sample sizes will be used: 25, 50, 100, and 200. The experiment was repeated nSim = 1000 times for each case. After that, the estimation process will be performed using both algorithms, and the comparison between them will be conducted using the Mean Squared Error (MSE) criterion. Equation (20) presents the formula used to calculate the MSE for each parameter.

Table 1. Different cases of the simulation experiments.

Cases	α	β
Case 1	0.5	2
Case 2	0.5	3
Case 3	1	2
Case 4	1	3
Case 5	2	2
Case 6	2	3
Case 7	3	2
Case 8	3	3

$$MSE(\hat{\theta}) = \frac{1}{nSim}\sum_{k=1}^{nSim}\left(\hat{\theta}_k - \theta\right)^2 \qquad \dots (20)$$

$$Estimate(\hat{\theta}) = \frac{1}{nSim}\sum_{k=1}^{nSim}\hat{\theta}_k \qquad \dots (21)$$

where θrepresents each parameter of the distribution, and $\hat{\theta}_j$ denotes the estimate of the parameter θ at iteration j.

Tables (2) – (9) present the results of the simulation experiments.

Table 2. MSE results of the different methods for the first case.

Sample Size	Methods	$\alpha = 0.5$		$\beta = 2$	
		Estimate	MSE	Estimate	MSE
n=25	NM	0.52431	0.00700	2.29187	0.93095
	AO	0.52377	0.00699	2.29054	0.92765
n=50	NM	0.51687	0.00433	2.16178	0.51614
	AO	0.51648	0.00431	2.15861	0.51304
n=100	NM	0.50934	0.00191	2.03515	0.18928
	AO	0.50871	0.00189	2.03498	0.18886
n=200	NM	0.50650	0.00097	2.05174	0.09582
	AO	0.50599	0.00097	2.05020	0.09388

The results presented in Table (2) indicate that both the Nelder–Mead algorithm and the Aquila Optimizer are capable of estimating the two parameters of the Fréchet distribution efficiently. The estimates are close to the true values, and the MSE values decrease as the sample size increases, which reflects the consistency of the estimators. A clear similarity in performance between the two methods can also be observed, with a slight advantage for the Aquila Optimizer in some cases, particularly in estimating the parameter β.

Table 3. MSE results of the different methods for the second case.

Sample Size	Methods	$\alpha = 0.5$		$\beta = 3$	
		Estimate	MSE	Estimate	MSE
n=25	NM	0.52438	0.00703	3.43842	2.08949
	AO	0.52376	0.00699	3.43602	2.08826
n=50	NM	0.51712	0.00432	3.23756	1.15123
	AO	0.51648	0.00431	3.23792	1.15380
n=100	NM	0.50929	0.00190	3.05281	0.42364
	AO	0.50871	0.00189	3.05253	0.42505
n=200	NM	0.50659	0.00099	3.07643	0.21442
	AO	0.50599	0.00097	3.07529	0.21128

Table (3) shows the continued efficiency of both the Nelder–Mead algorithm and the Aquila Optimizer in estimating the parameters of the Fréchet distribution, as the estimates remained close to the true values, with a clear decrease in the MSE values as the sample size increased. As in the first case, the performance of the two algorithms was very similar, with a slight advantage for the Aquila Optimizer in some values, indicating the stability and effectiveness of both methods in handling heavy-tailed data.

Table 4. MSE results of the different methods for the third case.

Sample Size	Methods	$\alpha = 1$		$\beta = 2$	
		Estimate	MSE	Estimate	MSE
n=25	NM	1.04798	0.02804	2.10008	0.18518
	AO	1.04753	0.02798	2.09924	0.18415
n=50	NM	1.03341	0.01736	2.05302	0.10830
	AO	1.03295	0.01722	2.05225	0.10816
n=100	NM	1.01799	0.00758	2.00625	0.04601
	AO	1.01743	0.00755	2.00601	0.04588
n=200	NM	1.01216	0.00390	2.02044	0.02232
	AO	1.01198	0.00389	2.01954	0.02227

Table 4 shows that both the Nelder–Mead algorithm and the Aquila Optimizer achieved accurate estimates of the parameters of the Fréchet distribution, as the estimated values were close to the true values, and the MSE values decreased noticeably as the sample size increased. The table also indicates the continued strong similarity between the two methods, with a slight advantage for the Aquila Optimizer in some cases, which further supports the reliability of both methods in parameter estimation.

Table 5. MSE results of the different methods for the fourth case.

Sample Size	Methods	$\alpha = 1$		$\beta = 3$	
		Estimate	MSE	Estimate	MSE
n=25	NM	1.04790	0.02801	3.15067	0.41502
	AO	1.04753	0.02797	3.14885	0.41429
n=50	NM	1.03348	0.01727	3.08076	0.24440
	AO	1.03296	0.01722	3.07841	0.24335
n=100	NM	1.01806	0.00760	3.01008	0.10298
	AO	1.01744	0.00755	3.00900	0.10322
n=200	NM	1.01275	0.00393	3.03011	0.05051
	AO	1.01198	0.00389	3.02931	0.05010

Table (5) shows that the Nelder–Mead algorithm and the Aquila Optimizer provided similar performance in estimating the parameters of the Fréchet distribution, as the estimates were close to the true values, with a gradual decrease in the MSE values as the sample size increased. A slight advantage of the Aquila Optimizer can also be observed in some values, which confirms its ability to achieve good accuracy alongside the stability of the Nelder–Mead algorithm across all sample sizes.

Table 6. MSE results of the different methods for the fifth case.

Sample Size	Methods	$\alpha = 2$		$\beta = 2$	
		Estimate	MSE	Estimate	MSE
n=25	NM	2.09729	0.11282	2.03919	0.04245
	AO	2.09505	0.11190	2.03898	0.04257
n=50	NM	2.06802	0.06926	2.02004	0.02555
	AO	2.06591	0.06891	2.01975	0.02557
n=100	NM	2.03636	0.03047	2.00049	0.01152
	AO	2.03487	0.03021	2.00015	0.01147
n=200	NM	2.02516	0.01560	2.00933	0.00555
	AO	2.02396	0.01558	2.00841	0.00545

Table (6) shows that both the Nelder–Mead algorithm and the Aquila Optimizer achieved accurate estimates of the parameters of the Fréchet distribution, as the estimated values were close to the true values, and the MSE values decreased as the sample size increased, reflecting an improvement in statistical accuracy. The results also indicate the continued similarity in performance between the two methods, with a slight advantage for the Aquila Optimizer in some cases, which confirms the effectiveness of both methods under this setting.

Table 7. MSE results of the different methods for the sixth case.

Sample Size	Methods	$\alpha = 2$		$\beta = 3$	
		Estimate	MSE	Estimate	MSE
n=25	NM	1.04790	0.02801	3.15067	0.41502
	AO	1.04753	0.02797	3.14885	0.41429
n=50	NM	1.03348	0.01727	3.08076	0.24440
	AO	1.03296	0.01722	3.07841	0.24335
n=100	NM	1.01806	0.00760	3.01008	0.10298
	AO	1.01744	0.00755	3.00900	0.10322
n=200	NM	1.01275	0.00393	3.03011	0.05051
	AO	1.01198	0.00389	3.02931	0.05010

Table (7) shows that both the Nelder–Mead algorithm and the Aquila Optimizer continued to provide accurate estimates of the parameters of the Fréchet distribution, with the estimated values remaining close to the true values and the MSE values decreasing as the sample size increased. The table also demonstrates the similarity in performance between the two methods across all sample sizes, with a slight advantage for the Aquila Optimizer in some results, confirming the efficiency and stability of both methods in estimating the distribution parameters.

Table 8. MSE results of the different methods for the seventh case.

Sample Size	Methods	$\alpha = 3$		$\beta = 2$	
		Estimate	MSE	Estimate	MSE
n=25	NM	2.09534	0.11154	3.06100	0.09614
	AO	2.09506	0.11192	3.05848	0.09578
n=50	NM	2.06645	0.06882	3.03154	0.05783
	AO	2.06593	0.06891	3.02959	0.05751
n=100	NM	2.03691	0.03105	3.00153	0.02607
	AO	2.03484	0.03021	3.00022	0.02580
n=200	NM	2.02495	0.01548	3.01456	0.01239
	AO	2.02396	0.01558	3.01262	0.01227

Table (8) shows the continued ability of both the Nelder–Mead algorithm and the Aquila Optimizer to estimate the parameters of the Fréchet distribution, as the estimates remained close to the true values, while the MSE values decreased with increasing sample size. The results also reveal a strong similarity between the two methods, with a slight advantage for the Aquila Optimizer in some cases, indicating the stability and accuracy of both methods under this experimental setting.

Table 9. MSE results of the different methods for the eighth case.

Sample Size	Methods	$\alpha = 3$		$\beta = 3$	
		Estimate	MSE	Estimate	MSE
n=25	NM	3.14690	0.25588	3.03684	0.04182
	AO	3.14260	0.25182	3.03552	0.04172
n=50	NM	3.10321	0.15603	3.01909	0.02516
	AO	3.09888	0.15502	3.01764	0.02520
n=100	NM	3.05929	0.06899	2.99996	0.01168
	AO	3.05231	0.06797	2.99918	0.01149
n=200	NM	3.04312	0.03521	3.00907	0.00547
	AO	3.03597	0.03505	3.00795	0.00542

Table (9) shows that both the Nelder–Mead algorithm and the Aquila Optimizer provided accurate estimates of the parameters of the Fréchet distribution, as the estimated values were close to the true values and the MSE values decreased as the sample size increased. A clear similarity in performance between the two methods also remained evident, with a slight advantage for the Aquila Optimizer in some cases, which further enhances the reliability of both algorithms in parameter estimation under this scenario.

The simulation results for all the cases under study indicate that both the Nelder–Mead algorithm and the Aquila Optimizer achieved strong performance in estimating the parameters of the Fréchet distribution across different true values of the shape and scale parameters. In all scenarios, the estimates were close to the true values, and a gradual decrease in the Mean Squared Error (MSE) was observed as the sample size increased, reflecting improved estimation accuracy and consistency in line with the theoretical properties of maximum likelihood estimators. Although the performance of the two methods was highly similar, the Aquila Optimizer showed a slight advantage in many cases, particularly in estimating the scale parameter and in handling heavy-tailed data, which confirms its ability to explore the search space efficiently and avoid local optima. Overall, the results highlight the potential of metaheuristic algorithms such as AO to serve as a promising alternative or supportive tool to conventional numerical methods in complex statistical estimation problems.

6. Application

The application will be conducted using real data taken from the thesis of Majid (2019), which represent the time to failure of restorative ceramic materials used in the teeth of patients attending the Specialized Center for Dentistry affiliated with the Karbala Health Directorate. The data are presented as follows:

Table 10. Real data.

0.14	0.16	0.19	0.24	0.99	1.24	2.20	2.17	3.23	9.17
0.70	0.16	0.27	1.13	1.14	2.28	1.17	1.19	1.14	4.27
0.80	0.21	0.25	1.80	1.51	2.60	2.11	2.17	2.60	9.24
0.60	0.15	0.28	0.28	1.40	1.90	1.30	1.50	3.20	3.50
0.10	0.11	0.18	0.23	2.90	3.90	4.29	3.50	4.27	9.80

The parameters of the Fréchet distribution were estimated using the Maximum Likelihood Estimation (MLE) method through the Aquila Optimizer, and the results were as follows:

Table 11. Parameter estimation results.

Parameters	Estimate
α	0.81882
β	0.53939

Based on the estimates presented in Table (11), the survival function of the distribution was estimated and is illustrated in Figure (2). It can be observed that the estimated curve (red line) shows good agreement with the actual curve (black line), which indicates the ability of the model to represent the survival pattern correctly over time. This result confirms the efficiency of the adopted estimation method and the accuracy of the Fréchet distribution in describing reliability and survival data.

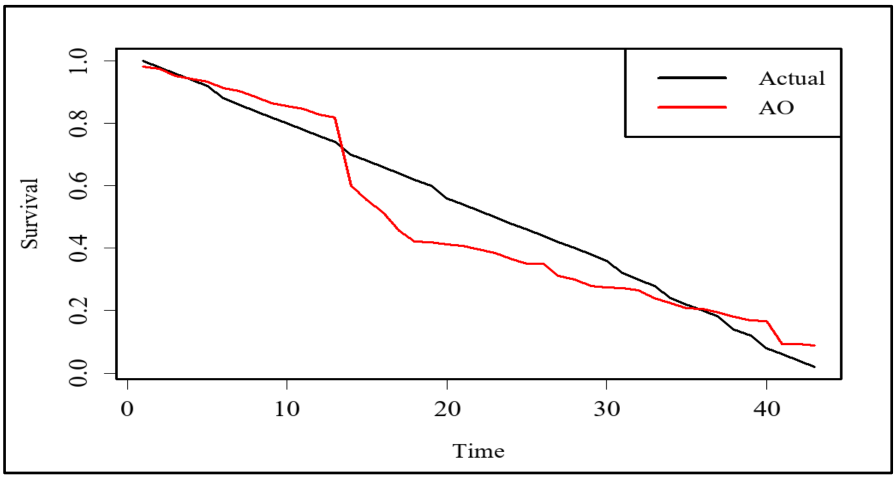

Figure 2. Plot of the survival function for the real data.

7. Conclusions

This study reached a set of important conclusions regarding the efficiency of the Fréchet distribution and the effectiveness of the Aquila Optimizer (AO) in estimating its parameters. The simulation experiments showed that both AO and traditional optimization methods such as Nelder–Mead are capable of producing accurate estimates of the distribution parameters, with a slight advantage in favor of AO in some cases, particularly for small samples. The results also demonstrated a noticeable decrease in MSE values as the sample size increased, reflecting the consistency of the estimation process and its capacity to improve when more data are available. These findings confirm that integrating metaheuristic optimization methods with extreme value distribution models represents an effective and applicable framework for modeling and analyzing risk, crisis, and rare-event data.

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