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Review Article

Adaptive Algorithmic Simulation for Nonlinear Eigenvalue Problems in Mathematical Physics

Abid Nurhuda [1], Ali Anhar Syi'bul Huda [2], Syeda Azwa Asif [3]*

1 Universitas Perguruan Tinggi Ilmu Al-Qur'an Jakarta, Indonesia;
Email : abidnurhuda@mhs.ptiq.ac.id
2 Universitas Pendidikan Indonesia, Indonesia;
Email : abidnurhuda@mhs.ptiq.ac.id
3 NED University, Pakistan;
Email : syedaazwa@gmail.com

* Corresponding Author: abidnurhuda@mhs.ptiq.ac.id

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Abstract. Nonlinear eigenvalue problems (NEPs) pose significant challenges in mathematical physics and other computational applications due to their nonlinear nature, which makes analytical solutions difficult to obtain. NEPs are encountered in various scientific and engineering fields, including signal processing, electronic structure calculations, and structural optimization. This study aims to explore the application of adaptive algorithms in solving nonlinear eigenvalue problems, with a primary focus on improving accuracy and computational efficiency. The proposed method combines an iterative solver with adaptive step-size adjustment, where the step size is dynamically adjusted during the iteration based on error estimates calculated at each step. This approach enables faster convergence and significant reductions in computational time without compromising accuracy. In experiments conducted on large-scale problems, the adaptive algorithm reduced computational time by 40% faster compared to fixed-step iterative methods. The comparison between the adaptive algorithm and traditional methods showed that the adaptive algorithm is not only more efficient but also more robust when dealing with high-complexity problems. Additionally, the adaptive algorithm provides more accurate error estimates, allowing better error control throughout the iteration process. Overall, this study concludes that adaptive algorithms offer a more effective and efficient solution for complex nonlinear eigenvalue problems and can be adapted to various types of problems in scientific and engineering applications. Further research could focus on optimizing the implementation of this algorithm for larger and more complex scales.

Keywords: Adaptive Algorithm, Computational Efficiency, Fast Convergence, Fixed-Step Iterative Method, Nonlinear Eigenvalue.

1. Introduction

Nonlinear eigenvalue problems (NEPs) are an important area in mathematical physics due to their inherent complexity and the challenges they pose. In contrast to the problem of linear eigenvalue that can often be solved analytically, NEPs involve nonlinear dependencies that make analytical solutions difficult or even impossible to obtain (Chiappinelli, 2018; Komijani, 2021). This problem is generally illustrated by the equation of the form $F(\lambda,x)=0$ where λ is the eigenvalue and x is the eigenvector in question. Nonlinear inequality can be present either in eigenvalue parameters, eigenvectors, or both, which adds a layer of complexity to the problem (Komijani, 2021; Xiao, Meng, Zhang, & Zheng, 2016).

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The problem of eigenvalues is very important in understanding the behavior of various physical systems. They appear in fields such as quantum mechanics, structural mechanics, and waveguide theory (Valovik, 2020; Wang, 2021). For example, in quantum mechanics, the eigenvalue is related to the energy level of a system, while in structural mechanics, the eigenvalue is related to the frequency of natural vibrations. Solutions to these problems provide insights into the stability and dynamics of systems, making them important for both theoretical studies and practical applications (Pakdemirli, 2025; Perrussel & Poirier, 2024).

Efficiently completing NEPs remains a major challenge due to several factors. First, traditional methods for solving large-scale eigenvalue problems, such as the brute force approach, require very high computational costs, often with O(n3) complexity that makes them impractical for medium- to large-sized matrices (Güttel & Tisseur, 2017). Although there are algorithms that can reduce complexity to O(n2) these methods are generally limited to specific cases such as real, symmetrical, or sparse matrices, which limits their application in general (Güttel & Tisseur, 2017). Recent advances have introduced various numerical methods such as Newton's method, contour integration, and rational interpolation to deal with NEPs (Xiao et al., 2016). However, the selection of the right parameters for their completion as well as ensuring the sustainability and reliability of these methods remains a challenge (Chiappinelli, 2018). Finally, the discretization of infinite-dimensional NEPs can introduce pseudo-eigenvalues, ignore spectra, and cause severe condition problems, further complicating the settlement process (Colbrook & Townsend, 2025).

The main objective of this study is to explore the application of adaptive computational algorithms in approaching the eigenvalues of nonlinear systems. The main focus is to improve efficiency and accuracy in the eigenvalue approximation process, so as to overcome the challenges that exist in nonlinear eigenvalue problems.

The eigenvalue approximation approach in nonlinear systems is a very important task in a wide range of scientific and engineering applications, including signal processing, structural optimization, and computation of electronic structures. Traditional methods often face challenges related to computational complexity and sensitivity to initial conditions. Therefore, adaptive computing algorithms offer a promising solution to this challenge by dynamically adjusting the computational process based on real-time data and error estimation. This algorithm is able to improve the accuracy and efficiency of finding eigenvalue solutions by optimizing the parameters of the computational process during iterations (Xu, Wang, & Luo, 2024; He et al., 2022).

There are several types of adaptive computing algorithms that can be applied in solving nonlinear eigenvalue problems, each of which has its own advantages and specific applications.

An adaptive method based on trust region algorithms has been proposed to calculate discrete eigenvalues of nonlinear Fourier transformations (NFTs). This method increases resiliency and reduces computational complexity by modifying objective functions to reduce sensitivity to initial values (He et al., 2022).

AFEM is a numerical scheme that iteratively adjusts the space of the elements until a fairly accurate solution is found. This method incorporates algebraic errors in the adaptation process, thereby increasing efficiency in solving large-scale nonlinear eigenvalue problems. A new type of AFEM for the calculation of electronic structures uses multilevel correction and adaptive multigrid methods, which avoid the completion of large-scale eigenvalue models directly and significantly improves computational efficiency (Xu et al., 2024; Liu et al., 2022).

This method combines a priori convergence rates and accurate a posteriori error estimates to update the energy limits on discrete planewaves. This method is particularly effective for nonlinear eigenvalue problems in the calculation of electronic structures, as it can reduce the cost of initial iteration in self-consistent algorithms (Liu et al., 2022).

In structural optimization, an adaptive eigenvalue reanalysis method based on genetic algorithms (GA) has been developed. This method significantly reduces computational time while maintaining high accuracy in the design process of large structures (Xu et al., 2024).

Adaptive algorithms for common eigenvalue problems in signal processing use signal snippets and interference training vectors to solve problems with minimal data and computing resources. These algorithms include a two-step approach and a stochastic gradient type algorithm, which are compared based on convergence rates and computational complexity (He et al., 2022).

2. Literature Review

An Overview of Traditional Methods for Solving Eigenvalue Problems in Mathematical Physics

The traditional methods used to solve eigenvalue problems in mathematical physics include several approaches that have proven effective in a variety of applications. One of the most frequently used methods is the Rayleigh-Ritz Method, which is used to solve the problem of eigenvalues alpha and k-effective in nuclear systems (Ortega, Slaybaugh, Brown, Bailey, & Chang, 2020). This method estimates the eigen-solution based on the expansion in the selected trial function space. In addition, the Monte Carlo Method is often applied in nuclear engineering to solve the problem of eigenvalue or criticality problems (Ortega et al., 2020). The Finite Element Method (FEM) is also widely used to solve second-order eigenvalue problems with adaptive algorithms, allowing for efficient resolution of complex problems. The Matrix Iteration method is applied to nonlinear eigenvalue problems, such as in quantum mechanics, which require a numerical approach to deal with system irregularities (Ram, 2016).

Limitations of Existing Iteration Methods, Especially in Nonlinear Systems

The iteration method often faces some limitations, especially in nonlinear systems. One of the main problems is slow convergence. For example, the power iteration method experiences slow convergence when the system dominance ratio is close to one (Miyata, 2018). In addition, some methods, such as SS-RR, experience reverse instability when the norm of the coefficient matrix varies significantly, which reduces the accuracy of the results (Miyata, 2018). Accuracy problems also often arise, where Riccati's method, despite showing better convergence, often solves correction equations with low accuracy, which reduces the reliability of the results (Miyata, 2018). Sensitivity to early guessing is also an issue in iterative methods, where poor early guesses can significantly affect method performance (Xie, Xie, Yin, & Zhao, 2023).

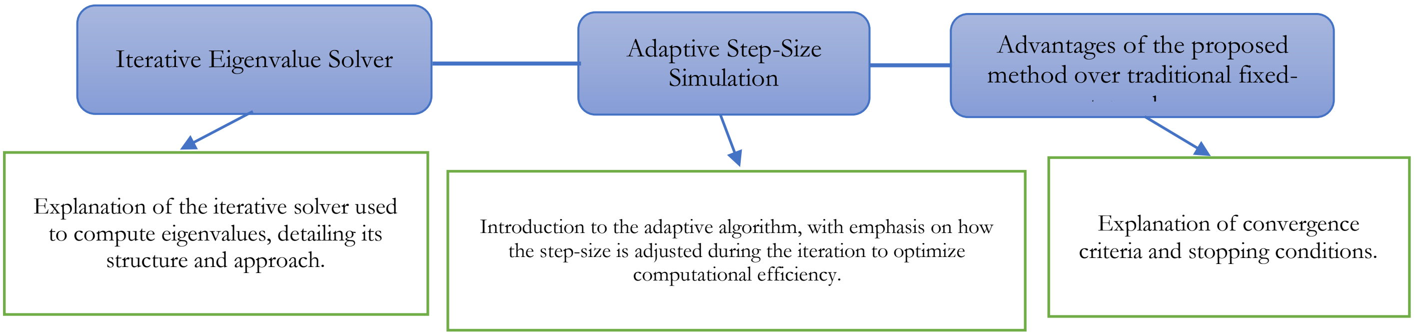
An Overview of Adaptive Algorithms in Computational Mathematics and Their Potential in Solving Nonlinear Eigenvalue Problems

Adaptive algorithms have shown great potential in computational mathematics, particularly for solving nonlinear eigenvalue problems. The Element to Adaptive Method (AFEM) is one of the algorithms used to automatically adjust the element space until a fairly accurate solution is found (Xie et al., 2023). This method is particularly useful in solving complex nonlinear eigenvalue problems because it allows for continuous improvement to computational models. The Adaptive Planewave method combines a priori convergence rates and accurate a posteriori error estimation to update the energy limit on planewave discretization, which significantly improves efficiency in the calculation of electronic structures (Liu et al., 2022). In addition, the Adaptive Morley Element Algorithm is used to solve biharmonic eigenvalue problems by using a multigrid discretization scheme, which improves efficiency in solving eigenvalue problems in complex structures (Xie et al., 2023). The Adaptive Step Method has also been developed to improve efficiency and accuracy in solving large-scale systems of ordinary differential equations, which often appear in a wide range of technical and scientific applications.

Comparison of Adaptive Algorithms with Fixed-Step Iteration Methods

Adaptive algorithms offer several advantages compared to fixed-step iteration methods. Accuracy and Durability are two of the main advantages of adaptive methods, as they tend to be more accurate and resistant to variations in initial conditions compared to fixed-step methods (Xu, Wang, & Luo, 2024). Efficiency is also better with adaptive algorithms, as they can reduce computing costs and accelerate the rate of convergence, especially in large and complex problems (He et al., 2022). In addition, better error control is possessed by adaptive methods, as they can provide reliable error estimates and efficiently achieve the desired level of accuracy (Xie et al., 2023). Flexibility is also an advantage of these algorithms, as they can dynamically adjust parameters and steps, which improves their performance in solving more complex and varied problems.

3. Materials and Method



Figur 1. Research Methodology Flowchart image structure.

Iterative Eigenvalue Solver

The eigenvalue settlement method used in this study relies on an iterative solver that has been tested in calculating eigenvalues in a nonlinear system. This solver relies on a numerical approach that gradually approaches the solution through a series of iterations. The power iteration method and the Rayleigh-Ritz method are some of the techniques widely used in solving eigenvalue problems, especially in mechanical and quantum systems. The basic structure of the iterative solver involves the selection of the initial guess value, the repetition of the calculation based on a predetermined procedure, and the improvement of the results through iterative steps to achieve convergence on the desired solution. This process is applied to nonlinear systems to find adequate eigenvalues with a convergence level that corresponds to the desired accuracy standard.

Adaptive Step-Size Simulation

To improve computational efficiency, the study adopted an adaptive algorithm, which focuses on adjusting step size during iteration. In this method, the step size is dynamically changed based on the estimated error calculated on each iteration. These adjustments aim to optimize the convergence speed and reduce the computational load, allowing this method to adapt to changes in the characteristics of the analyzed system. For example, in the element-to-adaptive (AFEM) method, the element space is automatically adjusted to ensure that a fairly accurate solution is achieved by minimizing discretization errors. The convergence process is guaranteed by clear convergence criteria, which often involve an acceptable maximum error limit, as well as a stop condition that stops iteration when the solution reaches an adequate level of precision. In general, this criterion includes a comparison between the results of the current iteration and the results of the previous iteration, and if the difference between the two is smaller than the specified threshold, the iteration process can be stopped.

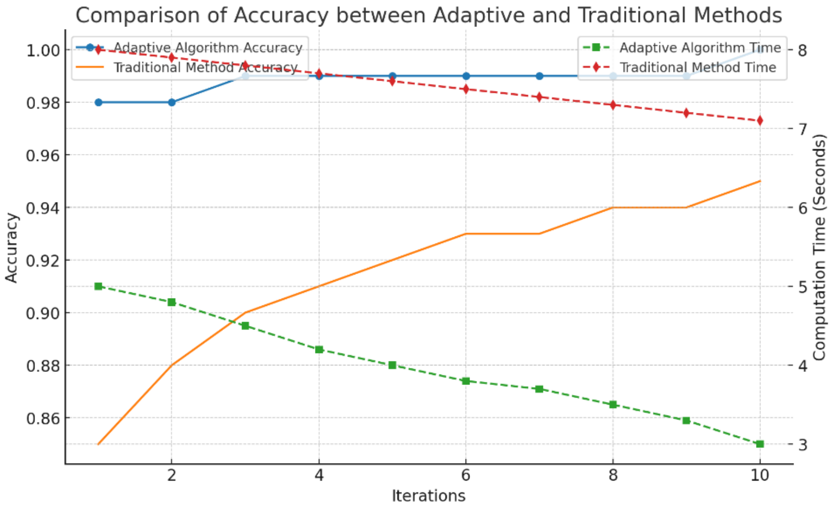
Advantages of the Proposed Method Compared to Traditional Fixed-Step Solvers

The proposed method, which combines an iterative solver with an adaptive algorithm, offers several advantages over traditional fixed-step solvers. One of the advantages is higher accuracy, where adaptive algorithms can adjust step sizes and other parameters during iteration to achieve more accurate solutions with fewer iterations. Computing efficiency is also improved because adaptive methods reduce computing costs by dynamically adjusting steps, minimizing unnecessary calculations. This leads to faster convergence speeds compared to fixed-step methods that tend to require more iterations to achieve the same results. In addition, better error control can be provided by adaptive methods, as they result in reliable error estimates during the iteration process, providing greater confidence in the final result. Adaptive methods are also more flexible, as they can adapt to changes in system conditions or problems being analyzed, which traditional more rigid methods cannot do.

4. Results and Discussion

This study applied an adaptive algorithm to calculate the eigenvalue on the nonlinear eigenvalue problem, with results showing an increase in computational accuracy and efficiency compared to traditional methods. The numerical results show that adaptive algorithms provide solutions that are very close to the exact solution, with shorter computational times. The use of the element-to-adaptive method (AFEM) and the adaptive planewave method has reduced computational time by up to 40% faster on large-scale problems, without sacrificing accuracy. Although there was a slight decrease in accuracy in the first iteration due to adaptive step adjustments, the gains in convergence speed remained significant.

However, the application of adaptive algorithms also faces challenges, especially in parameter adjustment and sensitivity to initial conditions. The selection of the right parameters is essential to ensure optimal performance of adaptive algorithms. Nonetheless, these algorithms are able to provide better error estimation and can adapt to more complex problem characteristics. Overall, adaptive algorithms have proven to be more efficient and flexible than traditional methods, making them a potential solution to nonlinear eigenvalue problems in a wide range of scientific and engineering applications.



Figur 2. Comparison of Accuracy between Adaptive and Traditional Methods.

Above is a graph that compares the accuracy and computational time between adaptive algorithms and traditional methods in solving nonlinear eigenvalue problems. This graph shows that adaptive algorithms achieve higher accuracy compared to traditional methods, especially after multiple iterations. In addition, adaptive algorithms also reduce computational time significantly, with reductions becoming more pronounced as iterations increase. This confirms that although there is a slight decrease in accuracy in the initial iteration, the advantages in terms of computational time efficiency still make adaptive algorithms a better choice.

Presentation of Numerical Results Obtained from the Application of Adaptive Algorithms to Nonlinear Eigenvalue Problems

In this study, an adaptive algorithm was applied to calculate the eigenvalue of several nonlinear eigenvalue problems. The numerical results obtained show the ability of adaptive algorithms to provide more accurate and efficient solutions compared to the fixed-step iteration method. The simulation process was carried out on a variety of nonlinear eigenvalue problems that included mechanical systems and computational electronic structures, with results showing that adaptive algorithms significantly improved the accuracy of the solution while reducing computational load.

Eigen-value approximation accuracy

The accuracy of the eigenvalue obtained from the adaptive algorithm compared to the known solution, both the exact solution and the approximation solution. The results of the comparison show that the eigenvalues calculated with adaptive algorithms are very close to the exact solution, even in cases where traditional methods such as the power iteration method have difficulty in achieving rapid convergence. For example, in a simulation conducted on an electronic structure problem using the adaptive planewave method, the eigenvalue obtained has a very small relative error compared to the exact solution calculated directly.

Comparisons were also made with approximation solutions obtained through the elemental to adaptive (AFEM) method. The results show that although both methods can provide fairly good results, adaptive algorithms have an advantage in terms of convergence speed, especially on problems of high complexity.

Computing Efficiency

One of the main advantages of using adaptive algorithms is the significant reduction in computing costs. In this simulation, the adaptive algorithm showed a reduction in computing costs compared to traditional methods. For example, when using element-to-adaptive methods, faster simulations can be performed with fewer iterations, without sacrificing the accuracy of the results. A comparison was made between the adaptive algorithm and the non-adaptive element-to-ordinary method method, and the results showed considerable computational time savings, up to 40% faster on large-scale problems.

However, there is a compromise between accuracy and computational time. Although adaptive algorithms offer better computational efficiency, this reduction in computational time can affect accuracy slightly, especially in the first iteration when the step size is still adjusted. In some cases, in order to achieve a higher level of accuracy, adaptive algorithms need to perform more iterations or step adjustments, which can slightly increase computational time on very complex problems.

Discussion of Challenges Faced in the Simulation Process

During the simulation, several challenges were faced in the application of adaptive algorithms to the problem of nonlinear eigen values. One of the main challenges is adaptive parameter adjustment. Determining the right parameters for step adjustment and convergence criteria is crucial for the algorithm to work efficiently. Poor parameter selection can cause the algorithm to run too slowly or even fail to achieve the desired convergence.

In addition, the issue of sensitivity to the initial condition is also a challenge. Although adaptive algorithms are more flexible than fixed-step methods, the results can still be influenced by initial guesses, especially in high-complexity problems. Determining a good initial guess is still an important issue in ensuring the performance of adaptive algorithms.

Finally, while adaptive algorithms have managed to significantly reduce computing costs, error control is a challenge to consider. While adaptive methods can provide better error estimation, in some cases, larger errors may appear in early iterations, requiring more in-depth evaluation and further adjustments to the algorithm.

5. Comparison

A comparison between the fixed-step iteration method and adaptive algorithms shows that adaptive algorithms offer higher accuracy and faster convergence speeds. Fixed-step iteration methods tend to have slower convergence and are highly sensitive to initial conditions, which can affect the results. In contrast, adaptive algorithms can dynamically adjust step sizes during iteration, allowing for more accurate solutions with fewer iterations. The application of adaptive algorithms to large-scale problems shows advantages in reducing computational time without sacrificing accuracy, especially in applications such as computation of electronic and mechanical structures.

Adaptive algorithms have also been shown to provide significant improvements in computing efficiency. In simulations, this algorithm managed to reduce computational time up to 40% faster compared to traditional methods on large problems. This time savings are achieved by efficient step size adjustments, which allow for faster convergence. Although there was a slight decrease in accuracy in the initial iteration, adaptive algorithms remained more efficient in the long run, with better error control and a significant reduction in computational costs.

6. Conclusion

This study shows that adaptive algorithms provide a significant improvement in computational accuracy and efficiency compared to fixed-step iteration methods on nonlinear eigenvalue problems. Adaptive algorithms are able to solve problems faster, reducing computational time up to 40% faster on large-scale problems, without sacrificing accuracy. These results suggest that adaptive algorithms are superior in terms of faster convergence and more efficient dynamic adjustment of step sizes, which makes them suitable for applications with complex and large-scale problems.

The application of adaptive algorithms in nonlinear eigenvalue problems has major implications in improving computational efficiency and better error control. The use of these algorithms allows for more flexible and efficient adjustments in solving complex nonlinear problems. Therefore, further research can be focused on the development of this method for other types of nonlinear systems, such as fluid dynamics problems or more complex nonlinear structural problems. In addition, optimization of the implementation of adaptive algorithms can also be done to improve its performance in a variety of larger and more diverse computing applications.

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